Allied Statistics: I

Unit - I

Theory of Probability:

Definitions:

1) A random experiment is one in which the exact outcome cannot be predicted before conducting the experiment.

Eg: (i) Rolling of a die
(ii) Tossing a coin.

2) A Sample space:

The set of all possible outcomes of a random experiment and is denoted by S.

3) A trial:

Each repetition of the experiment.

4) An event:

A Subset of the Sample Space S.

Eg:

Random Experiment	Sample Space	Some Events
(1) Tossing an unbiased coin once	$S = \{H, T\}$	The occurrence of head $\{H\}$ is an event. The occurrence of tail $\{T\}$ is another event.
(2) Rolling an unbiased die once	$S = \{1, 2, \cdots, 6\}$	$\{1,3,5\}$, $\{2,4,6\}$ & $\{6\}$ are some of the events.

5) Equally likely events:

Two or more events are said to be equally likely if each one of them has an equal chance of occurrence.

Eg: In tossing a coin, the occurrence of Head and the occurrence of tail are equally likely events.

6) Mutually exclusive events:

Two or more events are said to be mutually exclusive if the occurrence of one event prevents the occurrence of other events.

ie., Mutually exclusive events can't occur simultaneously.

Thus if A and B are two mutually exclusive events then $A \cap B = \phi$.

7) complementary events:

The set containing all the other outcomes which are not in E but in the sample space is called the complementary event of E.

It is denoted by $\bar{E}$. E and $\bar{E}$ are mutually exclusive events.

Eg:

In throwing a die, let $E = \{2, 4, 6\}$ be an event of getting a multiple of two.

Then the complementary of the event E is given by $\bar{E} = \{1, 3, 5\}$.

8) **Exhaustive events:**

Events $E_1, E_2, \ldots, E_n$ are exhaustive events if their union is the sample space (S).

9) **Sure event:**

The sample space of a Random experiment is called sure or certain event as any one of its element will surely occur in any trial of the experiment.

Eg:

getting one of 1, 2, 3, 4, 5, 6 in rolling a die is a sure event.

10) **Impossible event:**

An event which will not occur on any account is called an impossible event.

It is denoted by ϕ.

Eg: getting 7 in rolling a die once is an impossible event.

11) **Favourable outcomes:**

The outcomes corresponding to the occurrence of the desired event are called favourable outcomes of the event.

Eg: If E is an event of getting an odd number in rolling a die, then the outcomes 1, 3, 5 are favourable to the event E.

12) **classical definition of Probability** (or **Mathematical** (or) **A priori Definition** of **Probability**:

If a Sample Space contains 'n' outcomes and if 'm' of them are favourable to an event A, then, we write $n(S) = n$ and $n(A) = m$.

The probability of the event A, denoted by $P(A)$, is defined as the ratio of m to n.

i.e., $P(A) = \dfrac{\text{no. of outcomes favourable to A}}{\text{total no. of outcome}}$.

$$\therefore \boxed{P(A) = \frac{n(A)}{n(S)} = \frac{m}{n}}$$

Axiomatic Definition of Probability:

(i) $0 \le P(A) \le 1$

(ii) $P(S) = 1$

(iii) If A and B are mutually exclusively events then $P(A \cup B) = P(A) + P(B)$.

Note:

(i) $0 \le P(A) \le 1$

(ii) The probability of the Sure event is S.

ie., $P(S) = 1$

(iii) The probability of an impossible event is 0.

ie., $P(\varphi) = 0$.

(iv) $P(\text{not } A) = P(A')$ (or) $P(\bar{A})$

$$P(\bar{A}) = \frac{n-m}{n}$$

$$= \frac{n}{n} - \frac{m}{n}$$

$$= 1 - \frac{m}{n}$$

$$P(\bar{A}) = 1 - P(A)$$

$$\Rightarrow \boxed{P(A) + P(\bar{A}) = 1}$$

(v) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$.

Theorem: (1)

The Probability of the impossible event is zero. i.e., $P(\varphi)=0$

Proof:

The certain event S and the impossible event φ are mutually exclusive.

Hence $P(S\cup\varphi) = P(S) + P(\varphi)$

But $S\cup\varphi = S$

$$P(S) = P(S) + P(\varphi)$$

$$P(\varphi) = 0$$

Theorem: (2)

If $\bar{A}$ is the complementary event of A,

$P(\bar{A}) = 1 - P(A) \leq 1$.

Proof:

A and $\bar{A}$ are mutually exclusive events,

Such that $A\cup\bar{A} = S$

$\therefore P(A\cup\bar{A}) = P(S)$ ($\because$ we know that the probability of the Sample Space (S) in any experiment is 1)

$= 1$

(ie) $P(A) + P(\bar{A}) = 1$ ($\because$ A & $\bar{A}$ are mutually exclusive events)

$\therefore P(\bar{A}) = 1 - P(A)$

Since $P(A) \geq 0$, it follows that $P(\bar{A}) \leq 1$.

Theorem: (3) (Addition Theorem of Probability).

If A and B are any two events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

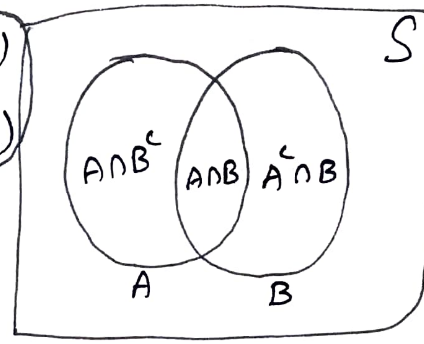
Proof:

For any two events A and B

$$P(A) = P(A \cap B) + P(A \cap B^c) \longrightarrow (1)$$

$$P(B) = P(A \cap B) + P(A^c \cap B) \longrightarrow (2)$$

$$(1)+(2) \Rightarrow P(A) + P(B) = \boxed{P(A \cap B)} + P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$$



$$= P(A \cap B) + P(A \cup B)$$

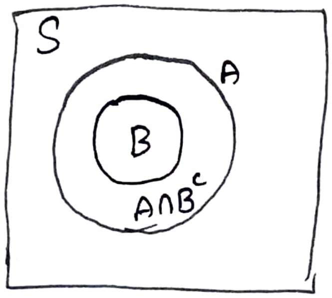
$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\leq P(A) + P(B)$$

Theorem: (4).

If $B \subset A$, $P(B) \leq P(A)$

Proof:



Here B and $A \cap B^c$ are mutually exclusive events such that $B \cup (A \cap B^c) = A$.

$$\therefore P[B \cup (A \cap B^c)] = P(A)$$

(ie) $P(B) + P(A \cap B^c) = P(A)$

$$P(B) \leq P(A)$$

Conditional Probability:

The conditional probability of an event B, assuming that the event A has happened, is denoted by $P(B/A)$ and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0.$$

Rewriting, we get $\boxed{P(A \cap B) = P(A) \times P(B/A)}$ is the product theorem of probability.

Properties:

(i) If $A \subset B$, $P(B/A) = 1$, since $A \cap B = A$

(ii) If $B \subset A$, $P(B/A) \geq P(B)$, since $A \cap B = B$,

and $\frac{P(B)}{P(A)} \geq P(B)$, as $P(A) \leq P(S) = 1$.

(iii) If A and B are mutually exclusive events, $P(B/A) = 0$, since $P(A \cap B) = 0$.

(iv) If $P(A) > P(B)$, $P(A/B) > P(B/A)$

(v) If $A_1 \subset A_2$, $P(A_1/B) \leq P(A_2/B)$

Statistical or A posteriori definition of probability:

Let a random experiment be repeated 'n' times and let an event A occur n_A times out of the n trials. The ratio of $\frac{n_A}{n}$ is called the relative frequency of the event A.

As n increases. $\frac{n_A}{n}$ shows a tendency to stabilise and to approach a constant value. This value denoted by P(A), is called the probability of the event A.

ie., $$P(A) = \lim_{n\to\infty} \frac{n_A}{n}.$$

Independent events:

If the events A and B are independent the product theorem takes the form

$$P(A\cap B) = P(A)\cdot P(B)$$

Conversely, If $P(A\cap B) = P(A) \times P(B)$, the events A and B are said to be independent.

Also, $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\cdot P(A_2)\cdots P(A_n)$, the events $A_1, A_2, \ldots, A_n$ are said to be totally independent.

Laws of Multiplication theorem on Probability:

(i) $P(A \cap B) = P(A) \cdot P(B|A)$, if $P(A) \neq 0$.

$P(A \cap B) = P(B) \cdot P(A|B)$, if $P(B) \neq 0$.

(ii) For independent events, $P(A \cap B) = P(A) \cdot P(B)$.

Note:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{P(A) \cdot P(B)}{P(A)}$$

$$= P(B)$$

$$P(B|A) = P(B)$$

Similarly, $P(A|B) = P(A)$

Laws of Addition theorem of Probability:

(i) Events are not mutually exclusive

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

(ii) Events are mutually exclusive

$$P(A \cup B) = P(A) + P(B) \quad (\because P(A \cap B) = 0)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

In general,

$$P(\Sigma A_i) = \Sigma P(A_i)$$

(iii) when events are independent,

If $P(A \cap B) = P(A) \cdot P(B)$

then $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$.

Theorem: (5)

If the events A and B are independent the events $\bar{A}$ and B (and similarly A and $\bar{B}$) are also independent.

Proof:

The events $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive events, such that

$(\bar{A} \cap B) \cup (A \cap B) = B$.

$$\therefore P(\bar{A} \cap B) + P(A \cap B) = P(B)$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A) \cdot P(B) \quad (\because A \,\&\, B \text{ are independent})$$

$$= P(B)(1 - P(A))$$

$$P(\bar{A} \cap B) = P(B) \cdot P(\bar{A}).$$

Theorem: (5) (ii)

If the events A and B are independent, So are $\bar{A}$ and $\bar{B}$.

Proof:

$$P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B})$$
$$= 1 - P(A\cup B)$$
$$= 1 - [P(A) + P(B) - P(A\cap B)]$$
$$= [1 - P(A)] - P(B) + P(A\cap B)$$
$$= P(\bar{A}) - P(B) + P(A)\cdot P(B)$$
($\because$ A & B are independent)
$$= P(\bar{A}) - P(B)\,[1 - P(A)]$$
$$= P(\bar{A}) - P(B)\cdot P(\bar{A})$$
$$= P(\bar{A})\,[1 - P(B)]$$
$$P(\bar{A}\cap\bar{B}) = P(\bar{A})\cdot P(\bar{B})$$

Hence proved.

Problems:

1) The probability of three students A, B, C solving a problem in statistics are $1/2$, $1/3$ & $1/4$. Any problem is given to all the three students. What is the probability that the problem will be solved.

(or)

what is the Probability that

(i) No one will solve the Problem.

(ii) Atleast one will Solve the Problem.

Soln:

Let P_1, P_2 and P_3 be the probability of three Students of Solving the Problem.

Here $P_1 = 1/2$, $P_2 = 1/3$, $P_3 = 1/4$

(i) P(No one will Solve the Problem) $= (1-P_1)\cdot(1-P_2)\cdot(1-P_3)$

$= (1-1/2)\cdot(1-1/3)\cdot(1-1/4)$

$= 1/2 \cdot 2/3 \cdot 3/4$

$= 1/4$ //

(ii) P(Atleast one will Solve the problem) $= 1 -$ P(No one will Solve the Problem)

$= 1 - 1/4$

$= 3/4$ //

2) If $P(A) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$, find $P(\bar{A} \cap \bar{B})$.

Soln:

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

$$= 1 - P(A \cup B) \longrightarrow (1)$$

W.K.T, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.4 + 0.7 - 0.3$$

$$P(A \cup B) = 0.8$$

$(1) \Rightarrow P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$$= 1 - 0.8$$

$$P(\bar{A} \cap \bar{B}) = 0.2$$

3) Two events A and B are such that $P(A) = 0.4$, $P(\bar{B}) = 0.3$ and $P(A \cap B) = 0.2$. Determine $P(A \cup B)$ and $P(\bar{A} \cap \bar{B})$.

Soln:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.4 + 0.7 - 0.2$$

$$P(A \cup B) = 0.9$$

$\because P(\bar{B}) = 1 - P(B)$

$0.3 = 1 - P(B)$

$P(B) = 1 - 0.3$

$P(B) = 0.7$

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) \quad \text{(De'Morgan's Law)}$$

$$= 1 - P(A \cup B)$$

$$= 1 - 0.9$$

$$\boxed{P(\bar{A} \cap \bar{B}) = 0.1}$$

4) Two balls are drawn at random with replacement from a box containing 15 red, 10 white balls. Calculate the probability that

(i) Both balls are Red.

(ii) 1st ball is Red, and 2nd ball is white

(iii) one of them is white and other is Red

Soln:

(i) Let A be the event, the first drawn ball is Red and B be the event, the second ball drawn is Red.

Then, as the balls drawn are with replacement,

$$\therefore P(A) = \frac{15}{25} = \frac{3}{5}$$

$$P(B) = \frac{15}{25} = \frac{3}{5}$$

As A and B are independent events

$$\therefore P(\text{both Red}) = P(A \text{ and } B)$$

$$= P(A \cap B)$$

$$= P(A) \cdot P(B)$$

$$= \frac{3}{5} \cdot \frac{3}{5}$$

$$= \frac{9}{25} \;//$$

(ii) Let A be the first ball drawn is Red. and B be the Second ball drawn is white.

$$P(A) = \frac{15}{25} = \frac{3}{5}, \quad P(B) = \frac{10}{25} = \frac{2}{5}$$

$$\begin{aligned} P(A \text{ and } B) &= P(A \cap B) \\ &= P(A) \cdot P(B) \\ &= \frac{3}{5} \cdot \frac{2}{5} \\ &= \frac{6}{25} \end{aligned}$$

(iii) Here,

$$P(W) = \frac{10}{25} = \frac{2}{5}, \quad P(R) = \frac{15}{25} = \frac{3}{5}$$

$$\begin{aligned} P(WR \text{ or } RW) &= P(WR) + P(RW) \\ &= P(W \cap R) + P(R \cap W) \\ &= P(W) \cdot P(R) + P(R) \cdot P(W) \\ &= \frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5} \\ &= \frac{6}{25} + \frac{6}{25} \end{aligned}$$

$$P(WR \text{ or } RW) = \frac{12}{25}$$

5) A card is drawn from a pack of 52 cards. So that each card is equally likely to be selected. Which of the following events are independent?

(i) A: The card drawn is a Spade.

B: The card drawn is an ace.

(ii) A: The card drawn is black.

B: The card drawn is King.

(iii) A: The card drawn is a king or a Queen.

B: The card drawn is a Queen or a Jack.

Proof:

(i) There are 13 cards of Spade in a pack.

$$P(A) = \frac{13}{52}$$

There are 4 Ace in a pack.

$$P(B) = \frac{4}{52}$$

$$A \cap B = \{\text{an Ace of Spade}\}$$

$$P(A \cap B) = \frac{1}{52} \mathbin{/\!/}$$

Now, $P(A) \cdot P(B) = \frac{13}{52} \cdot \frac{4}{52} = \frac{1}{52} \mathbin{/\!/}$

Since $P(A \cap B) = P(A) \cdot P(B)$

Here the events are independent.

(ii) There are 26 black cards in a pack.

$$P(A) = \frac{26}{52}$$

There are 4 King cards in a pack

$$P(B) = \frac{4}{52}$$

$A \cap B = \{$ 2 black King cards $\}$

$$P(A \cap B) = \frac{2}{52} \,//$$

Now, $P(A) \cdot P(B) = \frac{26}{52} \cdot \frac{4}{52} = \frac{2}{52} \,//$

Since $P(A \cap B) = P(A) \cdot P(B)$

Here, the events are independent.

(iii) There are 4 Kings and 4 Queens in a pack of cards.

$\therefore$ Total no. of outcomes favourable to the event A is 8.

$\therefore P(A) = 8/52 = 2/13$

Similarly, $P(B) = 8/52 = 2/13$

$\because P(A) = P(K \cup Q)$

$= P(K) + P(Q)$

$= \frac{4}{52} + \frac{4}{52} = \frac{8}{52} \,//$

$P(B) = P(Q \cup J)$

$= P(Q) + P(J)$

$= \frac{4}{52} + \frac{4}{52} = \frac{8}{52} \,//$

$A \cap B = \{4 \text{ Queens}\}$

$$P(A \cap B) = \frac{4}{52} = \frac{1}{13}$$

$$P(A) \cdot P(B) = \frac{2}{13} \cdot \frac{2}{13} = \frac{4}{169}$$

$$P(A \cap B) \neq P(A) \cdot P(B)$$

$\therefore$ The events A and B are not independent.

6) Find the chance of drawing 2 white balls in succession from a bag containing 5 Red and 7 white balls, the balls drawn not being repeated.

<u>Soln</u>: Let A be the event that the ball drawn is white in the first drawn.
B be the event that the ball drawn is white in the second drawn.

W.K.T, $P(A \cap B) = P(A) \cdot P(B/A)$ (Multiplication theorem).

Here, $P(A) = \frac{7C_1}{12C_1} = \frac{7}{12}$

$$P(B/A) = \frac{6C_1}{11C_1} = \frac{6}{11}$$

$$\therefore P(A \cap B) = \frac{7}{12} \cdot \frac{6}{11}$$

$$P(A \cap B) = 7/22$$

7) A bag contains 5 Red and 4 Black balls. Two balls are drawn one by one without replacement. What is the probability that the first ball is Red and the second ball is black.

Let A be the event of getting Red balls in the first drawn. B be the event of getting black balls in the second drawn.

$$P(A \cap B) = P(A) \cdot P(B/A)$$

Here, $P(A) = \frac{5C_1}{9C_1} = \frac{5}{9}$

$$P(B/A) = \frac{4C_1}{8C_1} = \frac{4}{8}$$

$$P(A \cap B) = \frac{5}{9} \cdot \frac{4}{8}$$

$$P(A \cap B) = \frac{5}{18} \text{ //}$$

8) In a shooting test, the probability of hitting the target is $\frac{1}{2}$ for A, $\frac{2}{3}$ for B and $\frac{3}{4}$ for C. If all of them fire at the target, find the probability that

(i) none of them hits the target and

(ii) Atleast one of them hits the target.

Soln:

Given, $P(A) = 1/2$

$P(B) = 2/3$

$P(C) = 3/4$

Let A, B and C be the events of hitting the target.

$P(\bar{A}) = 1 - P(A)$
$= 1 - 1/2$
$P(\bar{A}) = 1/2$ //

$P(\bar{B}) = 1 - P(B)$
$= 1 - 2/3$
$P(\bar{B}) = 1/3$ //

$P(\bar{C}) = 1 - P(C)$
$= 1 - 3/4$
$P(\bar{C}) = 1/4$ //

(i) P(none of them hits the target) $= P(\bar{A} \cap \bar{B} \cap \bar{C})$

$= P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$

$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}$

$= \frac{1}{24}$ //

(ii) P(Atleast one of them hits the target) = 1 − P(none of them hits the target)

$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

$= 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$

$= 1 - 1/24$

$= 23/24$ //

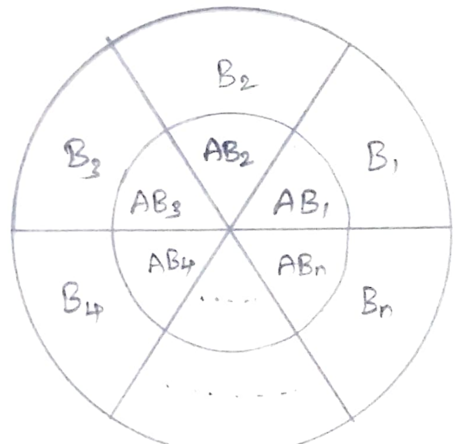
Theorem of Total Probability:

If $B_1, B_2, \ldots, B_n$ be a set of exhaustive and mutually exclusive events and A is another event associated with B_i then

$$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)$$

Proof:

The inner circle represent the event A. A can occur along with $B_1, B_2, \ldots, B_n$ that are exhaustive and mutually exclusive.



$\therefore A = AB_1, AB_2, \ldots, AB_n$ are also mutually exclusive.

$$\therefore P(A) = P\left[\sum_{i=1}^{n} AB_i\right]$$

$$= \sum_{i=1}^{n} P[AB_i] \quad \text{(since } AB_1, AB_2, \ldots, AB_n \text{ are mutually exclusive event)}$$

$$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P[A/B_i] \quad \text{(By product theorem)}$$

Baye's theorem:

If $B_1, B_2, \ldots, B_n$ be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with B_i, then

$$P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, \quad i=1,2,\dots,n$$

Proof:

$$P(B_i \cap A) = P(B_i) \cdot P(A/B_i) = P(A) \cdot P(B_i/A)$$

$$\Rightarrow P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{P(A)}$$

Here $P(A)$ is the total Probability

$$\therefore P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}, \quad i=1,2,\dots,n.$$

Problems:

1) An urn contains 10 white, 3 Black balls, while other urn contains 3 white, 5 Black balls. Two balls are drawn from the first urn and put into the 2nd urn. Then a ball is drawn from the later. what is the Probability that it is a white ball.

Soln:

The first urn contains 10 W, 3 B balls

The Second urn contains 3W, 5 B balls

The two balls drawn from the first urn may be

(i) Both white (event A_1)

(ii) Both Black (event A_2)

(iii) one white, one Black (event A_3)

$$\therefore P(A_1) = \frac{10C_2}{13C_2} = \frac{10\times 9/1\times 2}{13\times 12/1\times 2} = \frac{10\times 9}{13\times 12} = \boxed{\frac{15}{26}}.$$

$$P(A_2) = \frac{3C_2}{13C_2} = \frac{3\times 2/1\times 2}{13\times 2/1\times 2} = \frac{3\times 2}{13\times 2} = \boxed{\frac{1}{26}}.$$

$$P(A_3) = \frac{10C_1\times 3C_1}{13C_2} = \frac{10\times 3}{13\times 12/1\times 2} = \frac{10\times 3}{13\times 6} = \frac{5}{13} \text{ (or)}$$

$$= \boxed{\frac{10}{26}}$$

After the balls are transformed from 1st urn to 2nd urn. the 2nd urn will contain

(i) 5 white, 5 Black

(ii) 3 white, 7 Black

(iii) 4 white, 6 Black.

Let B be the event of drawing a white ball from the second urn.

$$P(B|A_1) = \frac{5C_1}{10C_1} = \frac{5}{10}$$

$$P(B|A_2) = \frac{3C_1}{10C_1} = \frac{3}{10}$$

$$P(B/A_3) = \frac{4C_1}{10C_1} = \frac{4}{10}$$

$$\therefore P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B/A_i)$$

$$= P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2) + P(A_3) \cdot P(B/A_3)$$

$$= \frac{15}{26} \cdot \frac{5}{10} + \frac{1}{26} \cdot \frac{3}{10} + \frac{10}{26} \cdot \frac{4}{10}$$

$$= \frac{75 + 3 + 40}{260}$$

$$= \frac{118}{260}$$

$$\boxed{P(B) = \frac{59}{130}}$$

2) Three Bags contains 6R, 4B ; 4R, 6B ; 5R, 5B balls respectively. one of the bags is Selected at random and a ball is drawn from it. If the ball drawn is Red, find the probability that it is drawn from the first bag.

Soln:

B_1 : Bag 1 is chosen

B_2 : Bag 2 is chosen

B_3 : Bag 3 is chosen

A : Ball drawn is red.

Since there are three Bags, one of them is chosen at random.

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$$P(A|B_1) = \frac{6C_1}{10C_1} = \frac{6}{10}$$

$$P(A|B_2) = \frac{4C_1}{10C_1} = \frac{4}{10}$$

$$P(A|B_3) = \frac{5C_1}{10C_1} = \frac{5}{10}$$

By Baye's theorem, the required Probability is

$$P(B_1/A) = \frac{P(B_1) \cdot P(A/B_1)}{\sum_{i=1}^{3} P(B_i) \cdot P(A|B_i)}$$

$$= \frac{1/3 \cdot 6/10}{\frac{1}{3} \cdot \frac{6}{10} + \frac{1}{3} \cdot \frac{4}{10} + \frac{1}{3} \cdot \frac{5}{10}}$$

$$= \frac{2/10}{\frac{6+4+5}{30}} = \frac{2}{10} \times \frac{30}{15} = \boxed{\frac{2}{5}}$$

3) Assume that a factory has two machines. Past records show that machine 1 produces 40% of the items. of output and machine 2 produces 60% of the items. Further, 5% of the items produced by machine 1 were defective and only 1% produced by machine 2

were defective. All the items so produced in the factory are stocked and an item of produced is selected at random.

(i) what is the probability that the item so chosen is defective. (To find: Total Probability)

(ii) If an item is selected at random is found to be defective. What is the probability that it was produced by machine 1. (To find : Baye's theorem)

Soln:

Let us define the following events.

E_1 : Item is Produced by machine 1.

E_2 : Item is Produced by machine 2.

E : Item is defective.

Then, $P(E_1) = \frac{40}{100}$, $P(E_2) = \frac{60}{100}$

$P(E/E_1) = \frac{5}{100}$, $P(E/E_2) = \frac{1}{100}$

(i) The Probability that an item selected at random is defective is $P(E)$.

By law of Total Probability,

$$P(E) = \sum_{i=1}^{2} P(E_i) \cdot P(E/E_i)$$

$$= P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)$$

$$= \frac{40}{100} \cdot \frac{5}{100} + \frac{60}{100} \cdot \frac{1}{100}$$

$$= \frac{200}{10000} + \frac{60}{10000}$$

$$= \frac{260}{10000}$$

$$\boxed{P(E) = 0.026}$$

(ii) Probability that item was produced by machine 1 given that item is defective is $P(E_1/E)$.

By Baye's theorem,

$$P(E_1/E) = \frac{P(E_1) \cdot P(E/E_1)}{\sum_{i=1}^{2} P(E_i) \cdot P(E/E_i)}$$

$$= \frac{40/100 \cdot 5/100}{26/1000}$$

$$P(E_1/E) = \frac{20/1000}{26/1000} = \frac{20}{26} = \boxed{0.769}$$ //

1·30

4) In a bolt factory machines A, B & C manufactures respectively 25 %, 35 % and 40 %. Of the total bolts of their outputs 5 %, 4 % and 2 % are respectively defective. A bolt is drawn at random from the total production.

(i) what is the probability that the bolt drawn is defective.

(ii) A bolt is drawn at random from the Production and found to be defective. What are the probabilities that it was manufactured by machines A, B & C.

Soln:

(i) Let us define the following events.

E_1 : A bolt manufactured by machine A.

E_2 : A bolt manufactured by machine B.

E_3 : A bolt manufactured by machine C.

E : The bolt is defective.

Then $P(E_1) = \frac{25}{100}$, $P(E_2) = \frac{35}{100}$, $P(E_3) = \frac{40}{100}$

$P(E/E_1) = 5/100$, $P(E/E_2) = 4/100$, $P(E/E_3) = \frac{2}{100}$

↓

(Probability of defective bolt produced by machine A)

By law of Total probability,

$$P(E) = \sum_{i=1}^{3} P(E_i) \cdot P(E/E_i)$$

$$= P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) + P(E_3) \cdot P(E/E_3)$$

$$= \frac{25}{100} \cdot \frac{5}{100} + \frac{35}{100} \cdot \frac{4}{100} + \frac{40}{100} \cdot \frac{2}{100}$$

$$= \frac{125}{10000} + \frac{140}{10000} + \frac{80}{10000}$$

$$= \frac{345}{10000}$$

$$P(E) = 0.0345$$

(ii) * P(defective bolt i=1 manufactured by machine A)

$$= P(E_1/E)$$

By the law of Baye's theorem,

$$P(E_1/E) = \frac{P(E_1) \cdot P(E/E_1)}{\sum_{i=1}^{3} P(E_i)\, P(E/E_i)}$$

$$= \frac{25/100 \cdot 5/100}{345/10000}$$

$$P(E_1/E) = \frac{125}{345} = 0.3623 \,//$$

* P(defective bolt i=2 manufactured by machine B)

$$= P(E_2/E)$$

$$P(E_2|E) = \frac{P(E_2) \cdot P(E/E_2)}{\sum_{i=1}^{3} P(E_i) \cdot P(E/E_i)}$$

$$= \frac{140/10,000}{345/10,000}$$

$$= \frac{140}{345} = 0.4058$$

* P(defective bolt i = 3 manufactured by machine C)

$$= P(E_3/E)$$

$$P(E_3|E) = \frac{P(E_3) \cdot P(E/E_3)}{\sum_{i=1}^{3} P(E_i) \cdot P(E/E_i)}$$

$$= \frac{80/10000}{345/10000}$$

$$= \frac{80}{345} = 0.2319$$

5) The first bag contains 3 white balls, 2 Red balls, and 4 Black balls. Second bag contains 2W, 3R and 5B Balls and third bag contains 3W, 4R and 2B Balls. One bag is chosen at random and from it three balls are drawn out of three balls 2 balls are white and 1 Red. What are the probabilities that they were taken from 1st bag, 2nd bag and 3rd bag.

Soln:

Let E_1: Bag 1 contains - 3W, 2R & 4B

E_2: Bag 2 contains - 2W, 3R & 5B

E_3: Bag 3 contains - 3W, 4R & 2B

Note that selection of bags are mutually exclusive event.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Let E be the event of drawing three balls which are 2W & 1R.

$$P(E/E_1) = \frac{3C_2 \times 2C_1}{9C_3} = \frac{\frac{3\times 2}{1\times 2} \times 2}{\frac{9\times 8\times 7}{1\times 2\times 3}} = \frac{1}{14}$$

$$P(E/E_2) = \frac{2C_2 \times 3C_1}{10C_3} = \frac{\frac{2\times 1}{1\times 2} \times 3}{\frac{10\times 9\times 8}{1\times 2\times 3}} = \frac{1}{40}$$

$$P(E/E_3) = \frac{3C_2 \times 4C_1}{9C_3} = \frac{\frac{3\times 2}{1\times 2} \times 4}{\frac{9\times 8\times 7}{1\times 2\times 3}} = \frac{1}{7}$$

Total Probability,

$$P(E) = \sum_{i=1}^{3} P(E_i) \cdot P(E/E_i)$$

$$= P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2) + P(E_3)\cdot P(E/E_3)$$

$$= \frac{1}{3}\left(\frac{1}{14}\right) + \frac{1}{3}\left(\frac{1}{40}\right) + \frac{1}{3}\left(\frac{1}{7}\right)$$

$$= \frac{1}{3}\left(\frac{1}{14} + \frac{1}{40} + \frac{1}{7}\right)$$

$$P(E) = 0.0798$$

(i) P(Ball Selected from 1st bag) = $P(E_1/E)$

$$P(E_1/E) = \frac{P(E_1) \cdot P(E/E_1)}{\sum_{i=1}^{n} P(E_i) \cdot P(E/E_i)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{14}}{0.0798}$$

$$P(E_1/E) = 0.2984 //$$

(ii) P(Ball selected from Second bag) = $P(E_2/E)$

$$P(E_2/E) = \frac{P(E_2) \cdot P(E/E_2)}{\sum_{i=1}^{3} P(E_i) \cdot P(E/E_i)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{40}}{0.0798}$$

$$P(E_2/E) = 0.1044 //$$

(iii) P(Ball Selected from third bag) = $P(E_3/E)$

$$P(E_3/E) = \frac{P(E_3) \cdot P(E/E_3)}{\sum_{i=1}^{3} P(E_i) \cdot P(E/E_i)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{7}}{0.0798}$$

$$P(E_3/E) = 0.5967 //$$

1.35

6) Let 5 men out of 100 and 25 women out of 1000 are colour blind. A colour blind person is chosen at random. What is the probability of his being male. (Assuming that males and females are in equal proportion).

Soln:

Let M denotes a person who is male.

Let F denotes a person who is female.

Let C denotes a person who is colour blind.

$$P(M) = P(F) = \frac{1}{2}$$

$$P(C/M) = \frac{5}{100}, \quad P(C/F) = \frac{25}{1000}$$

then, $$P(M/C) = \frac{P(M) \cdot P(C/M)}{P(M) \cdot P(C/M) + P(F) \cdot P(C/F)}$$

$$= \frac{\frac{1}{2} \cdot \frac{5}{100}}{\frac{1}{2} \cdot \frac{5}{100} + \frac{1}{2} \cdot \frac{25}{1000}}$$

$$= \frac{5/200}{\frac{5}{200} + \frac{25}{2000}} = \frac{1/40}{\frac{1}{40} + \frac{1}{80}} = \frac{1/40}{3/80}$$

$$P(M/C) = 0.667$$

1.36

7) Urn A contains 3W and 4B balls. Urn B contains 2W & 5B balls. 1 ball is transferred A to B and then 1 ball is drawn out of B. If this ball turns out to be white, find the probability that the transferred ball was white.

Soln:

Let us define the event as

B_1 : Ball transferred from urn A to B is white

$P(B_1) : 3/7$

B_2 : Ball transferred from urn A to B is Black

$P(B_2) = 4/7$

A: White Ball is drawn from urn B, so

$P(A/B_1) = 3/8$; $P(A/B_2) = 2/8$

Baye's theorem, $P(B_1/A) = \dfrac{P(B_1) \cdot P(A/B_1)}{\sum_{i=1}^{2} P(B_i) \cdot P(A/B_i)}$

$$= \frac{P(B_1) \cdot P(A/B_1)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)}$$

$$= \frac{3/7 \cdot 3/8}{3/7 \cdot 3/8 + 4/7 \cdot 2/8} = \frac{9/56}{9/56 + 8/56}$$

$$P(B_1/A) = \frac{9/56}{17/56} = 9/17 = \boxed{0.5294}$$ //

Unit - II

Random variables and Distribution functions:

Random Variables: (r.v) A real variable 'X' whose value is determined by the outcome of a random experiment is called a random Variable.

Example:

A random experiment consists of two tosses of a coin. Consider the random variable which is the number of heads.

Outcome :	HH	HT	TH	TT
Value of X :	2	1	1	0

Discrete random Variables:

A random variable which can assume only a countable number of real values is called a discrete random Variable.

Example:

Marks obtained in a test.

Probability mass function: (pmf)

If X is a discrete random variable taking atmost a countably infinite number of values $x_1, x_2, \ldots$ then

$P(X = x_i) = P_i$, the P_i is called the Probability function or pmf, provided P_i ($i = 1, 2, 3, \ldots$) satisfy the following conditions:

(i) $P_i \geq 0$ for all i and

(ii) $\sum_i P_i = 1$

The Probability distribution of Random Variable X:

The collection of Pairs $\{x_i, P_i\}$, $i = 1, 2, 3, \ldots$ is called the Probability distribution of Random variable X, which is Sometimes displayed in the form of table as given below:

$X = x_i$	x_1	x_2	x_3	$\ldots$	x_n
$P(X = x_i)$	P_1	P_2	P_3	$\ldots$	P_n

Continuous random variable:

If X is a r.v which can take all values (ie., infinite number of values) in an interval, then X is called a continuous r.v.

Probability density function: (pdf)

If X is a continuous r.v such that $P\{x - \frac{1}{2}dx \le x \le x + \frac{1}{2}dx\} = f(x)\,dx$, then $f(x)$ is called the pdf of X, provided $f(x)$ satisfies the following conditions,

(i) $f(x) \geq 0$ for all $x \in R$.

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Note:

* When x is continuous Random variable

$$P(X=a) = P(a \le X \le a) = \int_a^a f(x)\,dx = 0$$

* $$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$$
$$= \int_a^b f(x)\,dx$$

Cumulative distribution function: (cdf)

(or)

Distribution function of x:

If X is a r.v, discrete (or) continuous then $P(X \le x)$ is called the cumulative distribution function of x and denoted by $F(x)$.

If X is discrete,

$$\boxed{F(x) = \sum_{\substack{j \\ x_j \le x}} P_j}$$

If X is continuous,

$$\boxed{F(x) = P(-\infty < X \le x) = \int_{-\infty}^{x} f(x)\,dx}$$

Properties of the cumulative distribution function $F(x)$:

1) If F is the distribution function of the random variable X and if $a<b$, then $P(a<X\le b) = F(b) - F(a)$

2) If F is the distribution function of a one dimensional random variable x, then

   (i) $0 \le F(x) \le 1$.

   (ii) $F(x) \le F(y)$, If $x<y$.

3) If F is the distribution function of one dimensional random variable X then, $F(-\infty)=0$. and $F(\infty)=1$.

4) If X is a continuous random variable (2.5) then $\boxed{\frac{d}{dx} F(x) = f(x)}$ at all points where $F(x)$ is differentiable.

Problems:

1) A random variable X has the following probability function.

Values of $X = x$	0	1	2	3	4	5	6	7
$P(X=x) = p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Find k

(ii) Evaluate $P(X<6)$, $P(X\geq 6)$ and $P(0<X<5)$

(iii) Evaluate $P(1.5<X<4.5 / X>2)$

(iv) Find the minimum value of 'a' such that

$P(X\leq a) > \frac{1}{2}$

(v) Determine the distribution function of X.

Soln:

(i) We know that $\sum_{i=1}^{n} p_i = 1$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow 10k^2 + 9k - 1 = 0$$

$$10k(k+1) - 1(k+1) = 0$$

$(10k-1)(k+1)=0$

$10k-1=0$; $k+1=0$

$10k=1$; $\boxed{k=-1}$

$\boxed{k=1/10}$

Since $P(X)\geq 0$, the value $k=-1$ is not Permissible.

Hence, we have $\boxed{k=1/10}$

(ii) (a) $P(X<6) = P(X=0)+P(X=1)+\cdots+P(X=5)$

$= \frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}$

$= \frac{8}{10}+\frac{1}{100}$

$= \frac{80+1}{100} = \boxed{\frac{81}{100}}$

[Aliter Method:

$P(X<6) = 1-P(X\geq 6)$

$= 1-[P(X=6)+P(X=7)]$

$= 1-[2k^2+7k^2+k]$

$= 1-[9k^2+k]$

$= 1-\left[\frac{9}{100}+\frac{1}{10}\right] = 1-\left[\frac{9+10}{100}\right]$

$= 1-\frac{19}{100} = \frac{100-19}{100}$

$= \boxed{\frac{81}{100}}$]

(b) $P(X \geq 6) = 1 - P(X<6)$

$= 1 - \frac{81}{100}$

$= \frac{100-81}{100} = \boxed{\frac{19}{100}}$

(c) $P(0 < X < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 0 + k + 2k + 2k + 3k$

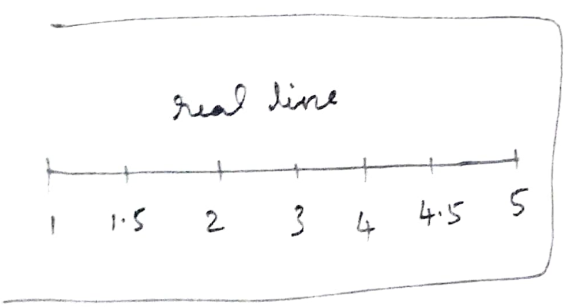
$= 8k$

$= \boxed{\frac{8}{10}}$

(iii) By using conditional probability,

$$\left\{ P(A|B) = \frac{P(A \cap B)}{P(B)} \right\}$$

$$P(1.5 < X < 4.5 \,/\, X > 2) = \frac{P(1.5 < X < 4.5 \cap X > 2)}{P(X>2)}$$



$$= \frac{P(2 < X < 4.5)}{P(X>2)}$$

$$= \frac{P(X=3) + P(X=4)}{1 - P(X \leq 2)}$$

$$= \frac{2k + 3k}{1 - (0 + k + 2k)}$$

$$= \frac{5k}{1-3k} = \frac{5/10}{1 - 3/10} = \frac{5/10}{7/10} = \boxed{\frac{5}{7}}$$

(v) The distribution function of X :

X	$P(X=x)$	$P(X \le x) = F(X)$
0	$P(0)=0$	$F(0) = P(0) = 0$
1	$P(1) = 1/10$	$F(1) = F(0) + P(1) = 1/10$
2	$P(2) = 2/10$	$F(2) = F(1) + P(2) = 3/10$
3	$P(3) = 2/10$	$F(3) = F(2) + P(3) = 5/10 = 1/2$
4	$P(4) = 3/10$	$F(4) = F(3) + P(4) = 8/10 > 1/2$
5	$P(5) = 1/100$	$F(5) = F(4) + P(5) = 81/100$
6	$P(6) = 2/100$	$F(6) = F(5) + P(6) = 83/100$
7	$P(7) = 17/100$	$F(7) = F(6) + P(7) = 100/100 = \boxed{1}$

(iv) The minimum value of $a = 4$ $\left(\because P(X \le a) > 1/2\right)$

H.W 2) A random variable X has the following pmf

X	0	1	2	3	4	5	6
$P(X=x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(i) Find k

(ii) $P(X<4)$, $P(X \ge 5)$, $P(3 < X \le 6)$

(iii) What is the smaller value of x for which $P(X \le x) > 1/2$?

Ans:

(i) $k = 1/49$ (ii) $P(X<4) = 16/49$

$P(X \geq 5) = 24/49$

$P(3 < X \leq 6) = 33/49$

(iii) The smallest value of $x = 4$ ($\because P(X \leq x) > 1/2$)

3) If the random variable X take the values 1, 2, 3 and 4 such that

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4).$$

Find the probability distribution and cumulative distribution function of x.

Soln:

Here X is a discrete r.v.

Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$2P(X=1) = k$	$3P(X=2) = k$	$P(X=3) = k$	$5P(X=4) = k$
$P(X=1) = k/2$	$P(X=2) = k/3$	$P(X=3) = k$	$P(X=4) = k/5$

Probability distribution function of X:

W.k.T $\sum_{i=1}^{n} P_i = 1$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{15k + 10k + 30k + 6k}{30} = 1$$

$$\frac{61k}{30} = 1$$

$$61k = 30$$

$$\Rightarrow \boxed{k = 30/61}$$

Distribution function of X:

x	$P(X=x)$	$F(x) = P(X \le x)$
1	$P(1) = k/2 = \frac{30}{61 \times 2} = \frac{15}{61}$	$F(1) = P(1) = k/2 = \frac{30}{61 \times 2} = \frac{15}{61}$
2	$P(2) = k/3 = \frac{30}{61 \times 3} = \frac{10}{61}$	$F(2) = F(1) + P(2) = \frac{15}{61} + \frac{10}{61} = \frac{25}{61}$
3	$P(3) = k = 30/61$	$F(3) = F(2) + P(3) = \frac{25}{61} + \frac{30}{61} = \frac{55}{61}$
4	$P(4) = k/5 = \frac{30}{61 \times 5} = \frac{6}{61}$	$F(4) = F(3) + P(4) = \frac{55}{61} + \frac{6}{61} = \frac{61}{61} = \boxed{1}$

4) Let X be a r.v such that $P(X=-2) = P(X=-1)$ $= P(X=1) = P(X=2)$ and $P(X<0) = P(X=0) = P(X>0)$. Determine the probability mass function of X and distribution function of X.

Soln:

Let $P(X=-2) = P(X=-1) = P(X=1) = P(X=2) = a$

Then $P(X<0) = P(X=0) = P(X>0) = 2a$

The Probability distribution is

X	-2	-1	0	1	2
$P(X=x)$	a	a	$2a$	a	a

$\therefore 6a = 1$ i.e., $\boxed{a = 1/6}$

$\therefore$ The probability mass function of X and distribution function of X are given by

X	$P(X=x)$	$F(x) = P(X \le x)$
-2	$P(-2) = 1/6$	$F(-2) = P(-2) = 1/6$
-1	$P(-1) = 1/6$	$F(-1) = F(-2) + P(-1) = \frac{2}{6}$
0	$P(0) = 2/6$	$F(0) = F(-1) + P(0) = 4/6$
1	$P(1) = 1/6$	$F(1) = F(0) + P(1) = 5/6$
2	$P(2) = 1/6$	$F(2) = F(1) + P(2) = 6/6 = \boxed{1}$

Pmf:

$X=x$	-2	-1	0	1	2
$P(X=x)$	$1/6$	$1/6$	$2/6$	$1/6$	$1/6$

cdf:

$X=x$	-2	-1	0	1	2
$F(x) = P(X \le x)$	$1/6$	$2/6$	$4/6$	$5/6$	1

5) If $P(X=x) = \frac{x}{15}$, $x = 1,2,3,4,5$ 2.12

find (i) $P(X=1 \text{ or } X=2)$, (ii) $P\left(\frac{1}{2} < X < \frac{5}{2} \,\middle|\, X>1\right)$

Soln:

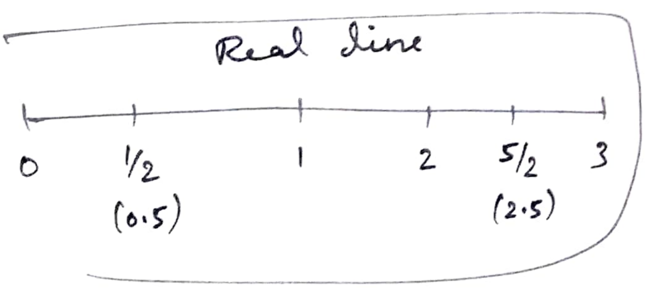
The prob. mass. fn. of X is

X	1	2	3	4	5
$P(X=x)$	$1/15$	$2/15$	$3/15$	$4/15$	$5/15$

(i) $P(X=1 \text{ or } X=2) = P(X=1) + P(X=2)$

$$= \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \boxed{\frac{1}{5}}$$

(ii) $P\left(\frac{1}{2} < X < \frac{5}{2} \,\middle|\, X>1\right) = \dfrac{P\left(\frac{1}{2} < X < \frac{5}{2} \cap X>1\right)}{P(X>1)}$



$$= \frac{P(X=2)}{P(X>1)}$$

$$= \frac{P(X=2)}{1-P(X \leq 1)} = \frac{2/15}{1 - 1/15}$$

$$= \boxed{\frac{1}{7}}$$

6) A coin is tossed twice. write the probability distribution of no. of heads.

Soln:

Let X be the random variable representing no. of heads and p be the probability of head in a single toss of a coin.

$\therefore p = 1/2 \quad , \quad q = 1/2$

($p \rightarrow$ success
$q \rightarrow$ failure)

Also, $X = 0, 1, 2$

Now $P(X=0) = q \cdot q = 1/4$

$P(X=1) = p \cdot q + q \cdot p = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$

$P(X=2) = p \cdot p = \frac{1}{4}$

So, the probability distribution is

$X = x$	0	1	2
$P(X=x)$	$1/4$	$2/4$	$1/4$

7) A bag contains 3 red and 4 white balls. Find the probability distribution of the number of red balls in 3 draws with replacement.

Soln:

Let X denote the number of red balls.

X takes values $0, 1, 2, 3$.

$P(X=0)$ = Probability of drawing no red ball

$= P(WWW)$

$= P(W) \cdot P(W) \cdot P(W)$

$= \frac{4}{7} \cdot \frac{4}{7} \cdot \frac{4}{7} = \boxed{\frac{64}{343}}$

$P(X=1)$ = Prob. of drawing 1 red ball

$= P(WWR \cup WRW \cup RWW)$

$= P(WWR) + P(WRW) + P(RWW)$

$= \frac{4}{7}\cdot\frac{4}{7}\cdot\frac{3}{7} + \frac{4}{7}\cdot\frac{3}{7}\cdot\frac{4}{7} + \frac{3}{7}\cdot\frac{4}{7}\cdot\frac{4}{7} = \boxed{\frac{144}{343}}$

$P(X=2)$ = Prob. of drawing 2 red balls

$= P[RRW \cup RWR \cup WRR]$

$= P(RRW) + P(RWR) + P(WRR)$

$= \frac{3}{7}\cdot\frac{3}{7}\cdot\frac{4}{7} + \frac{3}{7}\cdot\frac{4}{7}\cdot\frac{3}{7} + \frac{4}{7}\cdot\frac{3}{7}\cdot\frac{3}{7} = \boxed{\frac{108}{343}}$

$P(X=3)$ = Prob. of drawing 3 red balls

$= P(RRR)$

$= \frac{3}{7}\cdot\frac{3}{7}\cdot\frac{3}{7} = \boxed{\frac{27}{343}}$

pmf:

x	0	1	2	3
$P(x)$	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$

(ii) 8) Four bad oranges are mixed accidently with 16 good oranges. Find the probability distribution of the no. of bad oranges in a draw of 2 oranges.

Soln:

Let X denote the no. of bad oranges in a draw of 2 oranges.

So, X can assume the values 0, 1, 2.

Here, the no. of good oranges = 16

no. of bad oranges = 4

Now $P(X=0) = \dfrac{16C_2 \times 4C_0}{20C_2} = \dfrac{\frac{16\times15}{1\times\cancel{2}} \times 1}{\frac{20\times19}{1\times\cancel{2}}} = \dfrac{60}{95}$

$P(X=1) = \dfrac{16C_1 \times 4C_1}{20C_2} = \dfrac{16\times4}{\frac{20\times19}{1\times2}} = \dfrac{32}{95}$

$P(X=2) = \dfrac{16C_0 \times 4C_2}{20C_2} = \dfrac{1 \times \frac{4\times3}{1\times2}}{\frac{20\times19}{1\times2}} = \dfrac{3}{95}$

∴ The probability distribution is

$X=x$	0	1	2
$P(X=x)$	60/95	32/95	3/95

Problems :

1) A continuous random variable X follows the Probability law $f(x) = Ax^2$, $0 \le x \le 1$. Determine A and find the Probability that x lies b/w 0.2 and 0.5.

Soln :

(i) W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 Ax^2\,dx = 1$$

$$A\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$\frac{A}{3}[1-0] = 1$$

$$\frac{A}{3} = 1 \Rightarrow \boxed{A = 3}$$

(ii) W.K.T, $P(a < x < b) = \int_a^b f(x)\,dx$

$$P(0.2 < x < 0.5) = \int_{0.2}^{0.5} Ax^2\,dx$$

$$= A\left(\frac{x^3}{3}\right)_{0.2}^{0.5}$$

$$= \frac{\cancel{3}}{\cancel{3}}\left[(0.5)^3 - (0.2)^3\right]$$

$$= 0.125 - 0.008 = \boxed{0.117}$$

2) Verify whether $f(x) = \begin{cases} |x|, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

can be the pdf of a continuous R.V.

Soln:

To Prove: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-1}^{1} |x|\,dx = \int_{-1}^{1} x\,dx$$

$$= 2\int_{0}^{1} x\,dx$$

$$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_0^1$$

$$= 1^2 - 0^2$$

$$= \boxed{1}$$ Hence verified.

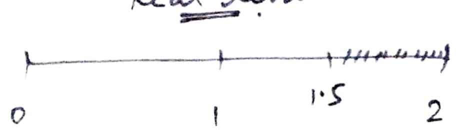
$\therefore$ The Given function is a p.d.f.

3) If the pdf of a random variable X is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$. Find $P(X > 1.5 \mid X > 1)$

Soln:

$$P(X > 1.5 \mid X > 1) = \frac{P(X > 1.5 \cap X > 1)}{P(X > 1)}$$

Real line



$$= \frac{P(1.5 < X < 2)}{P(X > 1)}$$

$$= \frac{\int_{1.5}^{2} x/2\,dx}{\int_{1}^{2} \frac{x}{2}\,dx}$$

$$= \frac{\cancel{\frac{1}{2}}\left[\frac{x^2}{\cancel{2}}\right]_{1.5}^{2}}{\cancel{\frac{1}{2}}\left[\frac{x^2}{\cancel{2}}\right]_{1}^{2}}$$

$$= \frac{2^2 - (1.5)^2}{2^2 - 1^2} = \frac{4 - 2.25}{4-1} = \frac{1.75}{3} = \boxed{0.5833}$$

4) A continuous R.V X had pdf $f(x) = K$, $0 \le x \le 1$. Determine the constant K and find $P(X \le 1/4)$

Soln:

(i) W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 K\,dx = 1$$

$$K\int_0^1 dx = 1$$

$$K[x]_0^1 = 1$$

$$K[1-0] = 1$$

$$\boxed{K = 1}$$

(ii) $P(X \le 1/4) = \int_0^{1/4} f(x)\,dx$

$$= \int_0^{1/4} K.\,dx$$

$$= [x]_0^{1/4}$$

$$= \boxed{1/4}$$

5) Given that the pdf of the R.V X is $f(x) = kx$, $0<x<1$, find k and $P(X>0.5)$ (2.19)

Soln:

(i) W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 kx \cdot dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$k\left[\frac{1}{2} - 0\right] = 1$$

$$\frac{k}{2} = 1 \Rightarrow \boxed{k=2}$$

(ii) $P(X>0.5) = \int_{0.5}^{1} f(x)\,dx$

$$= k\int_{0.5}^{1} x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_{0.5}^{1}$$

$$= 1^2 - (0.5)^2$$

$$= \boxed{0.75}$$

6) Suppose that X is a continuous R.V whose pdf is given by $f(x) = \begin{cases} c(4x - 2x^2), & 0<x<2 \\ 0, & \text{otherwise} \end{cases}$

Find c, $P(X>1)$.

(i) W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$c\int_0^2 (4x-2x^2)\,dx = 1$$

$$c\left[\int_0^2 4x\,dx - \int_0^2 2x^2\,dx\right] = 1$$

$$c\left[\frac{4x^2}{2}\right]_0^2 - \left[\frac{2x^3}{3}\right]_0^2 = 1$$

$$c\left[(8-0) - (16/3 - 0)\right] = 1$$

$$c\left[8 - 16/3\right] = 1$$

$$c\left[\frac{24-16}{3}\right] = 1$$

$$c\left[\frac{8}{3}\right] = 1$$

$$\boxed{c = 3/8}$$

(ii) $P(X>1) = \int_1^2 f(x)\,dx$

$$= c\int_1^2 (4x-2x^2)\,dx$$

$$= \frac{3}{8}\left[\frac{4x}{2} - \frac{2x^3}{3}\right]_1^2$$

$$= \frac{3}{8}\left[\left(\frac{4(4)}{2} - \frac{2(8)}{3}\right) - \left(\frac{4}{2} - \frac{2}{3}\right)\right]$$

$$= \frac{3}{8}\left[\left(8-\frac{16}{3}\right)-\left(\frac{12-4}{6}\right)\right]$$

$$= \frac{3}{8}\left[\frac{8}{3}-\frac{8}{6}\right]$$

$$= \frac{3}{8}\left[\frac{48-24}{18}\right]$$

$$= \frac{3}{8}\left[\frac{24}{18}\right]$$

$$= \boxed{1/2}$$ //

7) If $f(x) = \begin{cases} kxe^{-x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$

is the pdf of a R.V X. Find k.

Soln:

W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$\int_{0}^{\infty}(kxe^{-x})\,dx = 1$$

By Bernoulli's theorem,

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\left[\begin{array}{ll} u = kx & dv = e^{-x} \\ & v = -e^{-x} \\ u' = k & v_1 = e^{-x} \end{array}\right]$$

(+) (−)

$$\left(-kxe^{-x}\right)_0^{\infty} - \left(ke^{-x}\right)_0^{\infty} = 1$$

$$-k\left(\infty e^{-\infty}\right) - k\left(e^{-\infty} - e^{-0}\right) = 1$$

$$0 - k(0-1) = 1$$

$$\boxed{k = 1}$$

8) A continuous R.V X has pdf given by $f(x) = 3x^2,\ 0 \le x \le 1$ find k such that $P(X > k) = 0.5$.

Soln:

Given $P(X > k) = 0.5$

Now, $P(X > k) = 1 - P(X \le k)$

$$0.5 = 1 - \int_0^k f(x)\,dx$$

$$\int_0^k f(x)\,dx = 1 - 0.5$$

$$\int_0^k 3x^2\,dx = 0.5$$

$$\left[\frac{3x^3}{3}\right]_0^k = 0.5$$

$$k^3 = 0.5$$

$$k = \sqrt[3]{0.5}$$

$$\boxed{k = 0.7937}$$

9) Let X be a continuous RV with pdf

$$f(x) = \begin{cases} 1 & , 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$$

Find the cdf of the RV X.

Soln:

W.K.T, $F(x) = \int_{-\infty}^{x} f(x) \cdot dx$

(i) If $x < 0$,

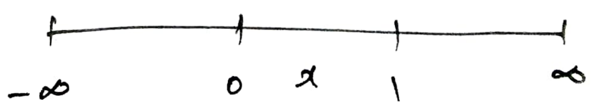
$$F(x) = \int_{-\infty}^{x} 0 \, dx = 0$$

$$\boxed{F(x) = 0, \quad x < 0}$$

(ii) If $0 < x < 1$,

$$F(x) = \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{x} f(x) \, dx$$

$$= 0 + \int_{0}^{x} 1 \, dx$$



$$= 1 \, (x)_0^x$$

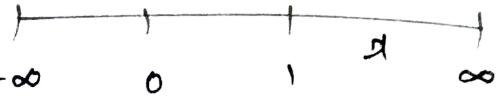
$$\boxed{F(x) = x, \quad 0 < x < 1}$$

(iii) If $x > 1$,

$$F(x) = \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{1} f(x) \, dx + \int_{1}^{x} f(x) \, dx$$

$$= 0 + \int_0^1 1\,dx + \int_1^x 0\,dx$$

$$= [x]_0^1$$



$$\boxed{F(x) = 1, \quad x > 1}$$

$$\therefore \boxed{F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 1, & x > 1 \end{cases}}$$

10) The pdf of a Continuous RV X is $f(x) = ke^{-|x|}$ find k and $F(x)$.

<u>Soln:</u>

W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} k e^{-|x|}\,dx = 1$$

$$k\left\{\int_{-\infty}^{0} e^{x}\,dx + \int_0^{\infty} e^{-x}\,dx\right\} = 1$$

$$k\left[\left(e^{x}\right)_{-\infty}^{0} + \left(\frac{e^{-x}}{-1}\right)_0^{\infty}\right] = 1$$

$$k\left[e^{0} - e^{-\infty} - e^{-\infty} + e^{0}\right] = 1$$

$$k[1 - 0 - 0 + 1] = 1$$

$$2k = 1$$

$$\boxed{k = 1/2}$$

(or)

$f(x) = ke^{-|x|}$

even or odd function ?

$\boxed{f(x) = ke^{-|x|} \rightarrow \text{even fn.}}$

$$\int_{-\infty}^{\infty} ke^{-|x|}\,dx = 2\int_0^{\infty} ke^{-x}\,dx$$

$$1 = 2k\int_0^{\infty} e^{-x}\,dx$$

$$1 = 2k\left(\frac{e^{-x}}{-1}\right)_0^{\infty}$$

$$1 = -2k\left(e^{-\infty} - e^{-0}\right)$$

$$1 = -2k(0 - 1)$$

$$1 = 2k$$

$$2k = 1 \Rightarrow \boxed{k = 1/2}$$

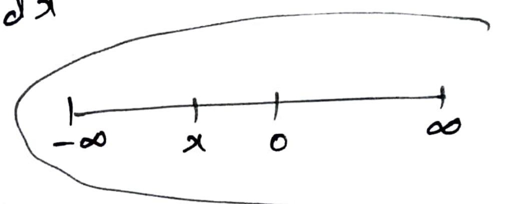
$\therefore$ The pdf is $f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$ (2.25)

So, $f(x) = \begin{cases} \frac{1}{2} e^{x}, & -\infty < x < 0 \\ \frac{1}{2} e^{-x}, & 0 < x < \infty \end{cases}$

To find: Cumulative distribution function (cdf)

(i) $x < 0$,

W.K.T, $F(x) = \int_{-\infty}^{x} f(x)\, dx$



$$= \int_{-\infty}^{x} \frac{1}{2} e^{x}\, dx$$

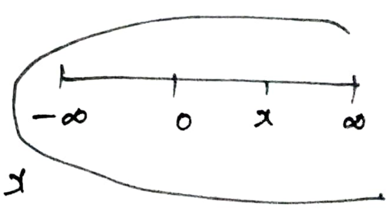
$$= \frac{1}{2} \left(e^{x}\right)_{-\infty}^{x}$$

$$= \frac{1}{2} \left[e^{x} - e^{-\infty}\right]$$

$$= \frac{e^{x}}{2}$$

$$\boxed{F(x) = \frac{1}{2} e^{x}, \quad x < 0}$$

(ii) $x > 0$, $F(x) = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{x} f(x)\, dx$



$$= \int_{-\infty}^{0} \frac{1}{2} e^{x}\, dx + \int_{0}^{x} \frac{1}{2} e^{-x}\, dx$$

$$= \frac{1}{2} \left(e^{x}\right)_{-\infty}^{0} + \frac{1}{2} \left(-e^{-x}\right)_{0}^{x}$$

$$= \frac{1}{2}\left(e^{0} - e^{-\infty}\right) - \frac{1}{2}\left(e^{-x} - e^{0}\right)$$

$$= \frac{1}{2}(1) - \frac{1}{2}\left(e^{-x} - 1\right)$$

$$= \frac{1}{2} - \frac{e^{-x}}{2} + \frac{1}{2}$$

$$= \frac{1}{2}\left(1 - e^{-x} + 1\right)$$

$$\boxed{F(x) = \frac{1}{2}\left(2 - e^{-x}\right), \quad x > 0}$$

$$F(x) = \begin{cases} \frac{1}{2} e^{x}, & x < 0 \\ \frac{1}{2}\left(2 - e^{-x}\right), & x > 0 \end{cases}$$

11) A continuous R.V X that can assume any value b/w $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1+x)$, (i) Find k (ii) $P(X < 4)$

Soln:

(i) W.K.T,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{2}^{5} k(1+x)\,dx = 1$$

$$k\left[\int_{2}^{5} dx + \int_{2}^{5} x\,dx\right] = 1$$

$$k\left[(x)_2^5 + \left(\frac{x^2}{2}\right)_2^5\right] = 1$$

$$k\left[(5-2) + \frac{5^2}{2} - \frac{2^2}{2}\right] = 1$$

$$k\left[3 + \frac{25}{2} - \frac{4}{2}\right] = 1$$

$$k\left[\frac{6+25-4}{2}\right] = 1$$

$$k\left[\frac{27}{2}\right] = 1$$

$$\boxed{k = \frac{2}{27}}$$

(ii) $P(X<4) = \int_2^4 f(x)\,dx$

$$= k\int_2^4 (1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4 = \frac{2}{27}\left[\left(4 + \frac{4^2}{2}\right) - \left(2 + \frac{2^2}{2}\right)\right]$$

$$= \frac{2}{27}\left[4 + \frac{16}{2} - 2 - \frac{4}{2}\right]$$

$$= \frac{2}{27}\left[4 + 8 - 2 - 2\right]$$

$$= \frac{2}{27}(8) = \boxed{\frac{16}{27}}$$

12) If pdf of a R.V X is given by

$$f(x) = \begin{cases} 1/4, & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find $P[|x| > 1]$

Soln:

$$P[|x|>1] = 1 - P[|x| \le 1]$$

$$= 1 - P(-1 < x < 1) = 1 - \int_{-1}^{1} f(x)\,dx$$

$$= 1 - \int_{-1}^{1} 1/4\,dx$$

$$= 1 - 2\int_{0}^{1} 1/4\,dx$$

$$= 1 - 2\left[1/4\,x\right]_0^1 = 1 - 2\left[1/4\right]\left[x\right]_0^1$$

$$= 1 - \frac{2}{4}\left[1 - 0\right] = 1 - \frac{2}{4} = 1 - 1/2 = \boxed{1/2}$$

$$\therefore \boxed{P[|x|>1] = 1/2}$$

13) If the density function of continuous R.V X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find a

(ii) The cdf of x

(iii) $P(x \leq 1.5)$

Soln:

(i) W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$\int_0^1 ax.\,dx + \int_1^2 a.\,dx + \int_2^3 3a - ax = 1$$

$$a\left[\left(\frac{x^2}{2}\right)_0^1 + \left(x\right)_1^2 + \left(3x - \frac{x^2}{2}\right)_2^3\right] = 1$$

$$a\left[\frac{1}{2} + (2-1) + \left(9 - \frac{9}{2} - 6 + \frac{4}{2}\right)\right] = 1$$

$$a\left[\frac{3}{2} + 5 - \frac{9}{2}\right] = 1$$

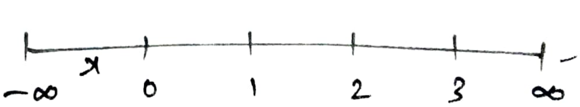
$$a[2] = 1$$

$$\Rightarrow \boxed{a = 1/2}$$

(ii) C.d.f of x:

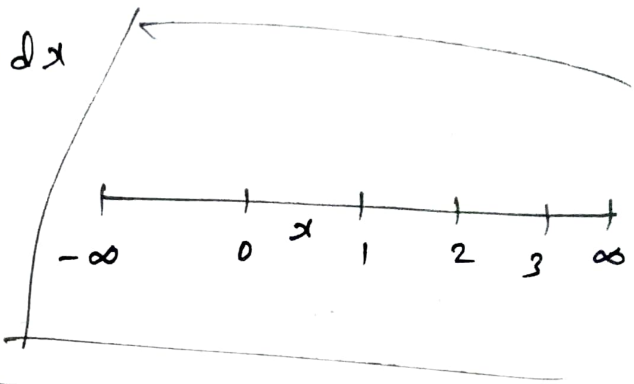
(a) If $x < 0$,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = 0$$



(b) If $0 \le x \le 1$,

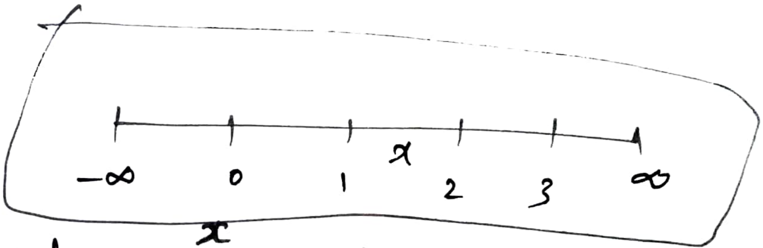
$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$



$$= 0 + \int_{0}^{x} ax\,dx$$

$$= a\left(\frac{x^2}{2}\right)_0^x = \frac{ax^2}{2} = \boxed{\frac{1}{4}x^2}$$

(c) If $1 \le x \le 2$



$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{x} a\,dx$$

$$= \left(\frac{ax^2}{2}\right)_0^1 + (ax)_1^x$$

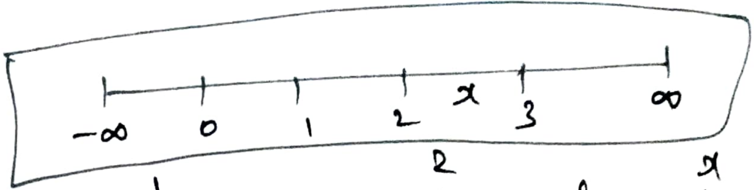
$$= \frac{a}{2} + ax - a$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$= \boxed{\frac{x}{2} - \frac{1}{4}}$$

(2.31)

(d) If $2 \le x \le 3$,



$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a.\,dx + \int_{2}^{x} (3a - ax)\,dx$$

$$= \left(\frac{ax^2}{2}\right)_0^1 + (ax)_1^2 + \left(3ax - \frac{ax^2}{2}\right)_2^x$$

$$= \frac{a}{2} + 2a - a + 3ax - \frac{ax^2}{2} - 6a + 2a$$

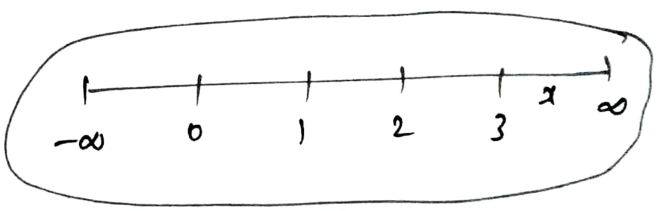
$$= -\frac{5a}{2} + 3ax - \frac{ax^2}{2}$$

$$= -\frac{5}{4} + \frac{3x}{2} - \frac{x^2}{4}$$

$$= -\frac{1}{4}(x^2 - 6x + 5) \,//$$

(e) If $x > 3$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$



$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (3a - ax)\,dx + \int_{3}^{x} 0\,dx$$

$$= \left(\frac{ax^2}{2}\right)_0^1 + (ax)_1^2 + \left(3ax - \frac{ax^2}{2}\right)_2^3$$

$$= \frac{a}{2} + (2a - a) + \left(9a - \frac{9a}{2} - 6a + 2a\right)$$

$$= \frac{a}{2} + a + \frac{9a}{2}$$

$$= -\frac{8a}{2} + 6a$$

$$= -4a + 6a$$

$$= 2a$$

$$= 1$$

The cdf is $F(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{4} & , 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{4} & , 1 \le x \le 2 \\ -\frac{1}{4}(x^2 - 6x + 5) & , 2 \le x \le 3 \\ 1 & , x > 3 \end{cases}$

(iii) $P(X \le 1.5) = \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx$

$$= \int_0^1 ax\,dx + \int_1^{1.5} a\,dx$$

$$= \left(\frac{ax^2}{2}\right)_0^1 + (ax)_1^{1.5}$$

$$= \frac{a}{2} + 1.5a - a$$

$$= \frac{a}{2} + \frac{a}{2}$$

$$= a$$

$$\boxed{P(x \le 1.5) = 1/2} \qquad (\because a = 1/2)$$

14) If $f(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(i) S.T $f(x)$ is a p.d.f of continuous R.V X.

(ii) Find its distribution function $F(x)$.

Soln:

(i) To Prove: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$L.H.S = \int_0^{\infty} x e^{-x^2/2}\,dx$$

put $\boxed{t = x^2/2}$

$$dt = \frac{2x}{2}\,dx \Rightarrow \boxed{dt = x\,dx}$$

When $x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$$L.H.S = \int_0^{\infty} e^{-t}\,dt$$

$$= \left(\frac{e^{-t}}{-1} \right)_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0} \right]$$

$$= -[0-1]$$

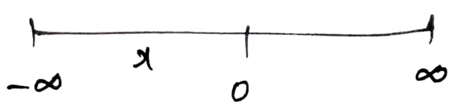
$$= 1$$

$$= R.H.S$$

$\therefore$ The given $f(x)$ is a p.d.f.

(ii) (a) If $x < 0$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$



$$= \int_{-\infty}^{x} 0 . dx$$

$$F(x) = 0$$

(b) $x > 0$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} x e^{-x^2/2}\,dx$$

$\boxed{t = x^2/2}$, when $\boxed{\begin{matrix} x = 0, & t = 0 \\ x = x, & t = x^2/2 \end{matrix}}$

$dt = \frac{2x\,dx}{2} \Rightarrow \boxed{dt = x\,dx}$

$$F(x) = \int_{t=0}^{t=x^2/2} e^{-t}\, dt$$

$$= \left(-e^{-t}\right)_0^{x^2/2}$$

$$= -e^{-x^2/2} + e^0$$

$$= -e^{-x^2/2} + 1$$

$$F(x) = 1 - e^{-x^2/2}, \quad x > 0$$

(2.35)

$$\boxed{F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x^2/2}, & \text{if } x > 0 \end{cases}}$$

15) If the c.d.f is given by $F(x) = \begin{cases} 1 - 4/x^2, & x > 2 \\ 0, & x \leq 2 \end{cases}$

Find (i) $P(X < 3)$, (ii) $P(4 < X < 5)$

(iii) $P(X \geq 3)$

Soln:

(i) $P(X < x) = F(x)$

$P(X < 3) = F(3)$

$= 1 - \frac{4}{9}$

$$\boxed{P(X < 3) = 5/9}$$

(ii) W.K.T, $P(a < x < b) = F(b) - F(a)$

$$P(4 < x < 5) = \left(1 - 4/5^2\right) - \left(1 - 4/4^2\right)$$

$$= \left(1 - 4/25\right) - \left(1 - 4/16\right)$$

$$= 1 - \frac{4}{25} - 1 + \frac{4}{16}$$

$$= \frac{4}{16} - \frac{4}{25}$$

$$= \frac{9}{100}$$

$$\boxed{P(4 < X < 5) = 0.09}$$

(iii) $P(X \geq 3) = 1 - P(X < 3)$

$$= 1 - 5/9$$

$$\boxed{P(X \geq 3) = 4/9}$$

16) A continuous R.V X has the p.d.f

$f(x) = 3x^2, \quad 0 \leq x \leq 1.$

(i) Find a, if $P(X \leq a) = P(X > a)$

(ii) Find b, if $P(X > b) = 0.05$

Soln:

(i) Given, $P(X \le a) = P(X > a)$

$\Rightarrow P(X \le a) = 1 - P(X \le a)$

$\Rightarrow 2P(X \le a) = 1$

$\Rightarrow P(X \le a) = 1/2$

$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$

$\Rightarrow \int_0^a 3x^2\,dx = \frac{1}{2}$

$\Rightarrow \cancel{3}\left(\frac{x^3}{\cancel{3}}\right)_0^a = \frac{1}{2}$

$\Rightarrow a^3 - 0^3 = 1/2$

$\Rightarrow a^3 = 1/2$

$\Rightarrow \boxed{a = \sqrt[3]{1/2}}$ //

(or)

$\boxed{a = (1/2)^{1/3}}$

(ii) Given, $P(X > b) = 0.05$

$\Rightarrow \int_b^1 f(x)\,dx = 0.05$

$\Rightarrow \int_b^1 3x^2\,dx = 0.05$

$\Rightarrow \left(\frac{\cancel{3}x^3}{\cancel{3}}\right)_b^1 = 0.05$

$\Rightarrow 1^3 - b^3 = 0.05$

$\Rightarrow 1 - b^3 = 0.05$

$\Rightarrow b^3 = 1 - 0.05$

$\Rightarrow b^3 = 0.95 = \frac{\cancel{95}^{19}}{\cancel{100}_{20}}$

$\Rightarrow b^3 = \frac{19}{20}$

$\Rightarrow \boxed{b = \left(\frac{19}{20}\right)^{1/3}}$ //

2.38

17) A continuous R.V X, has the distribution function $F(x) = \begin{cases} 0, & x < 1 \\ k(x-1)^4, & 1 \le x \le 3 \\ 1, & x > 3 \end{cases}$

find k, Probability density function $f(x)$, $P(X<2)$

<u>Soln:</u>

W.K.T, $P[X \le x] = F(x)$

$$P[X < 2] = F(2)$$
$$= k(2-1)^4$$
$$= k(1)^4 = k \longrightarrow (1)$$

To <u>find</u> k: <u>First to find the p.d.f</u>

W.K.T, $f(x) = \dfrac{d}{dx} F(x)$

$$= \frac{d}{dx}\left[k(x-1)^4\right] = k \cdot 4(x-1)^3$$

$$\therefore \boxed{f(x) = \begin{cases} 0, & x \le 1 \\ 4k(x-1)^3, & 1 < x \le 3 \\ 0, & x > 3 \end{cases}}$$

(or)

$$\boxed{f(x) = \begin{cases} 4k(x-1)^3, & 1 < x \le 3 \\ 0, & \text{otherwise} \end{cases}}$$

W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{1}^{3} 4k(x-1)^3\,dx = 1$$

$$4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1 \Rightarrow k\left[(x-1)^4\right]_1^3 = 1$$

$$\Rightarrow k\left[(3-1)^4 - (1-1)^4\right] = 1$$

$$\Rightarrow k\left[2^4 - 0\right] = 1$$

$$\Rightarrow 16k = 1$$

$$\Rightarrow \boxed{k = 1/16}$$

$P(X<2) = F(2) = k \quad (\text{by } (1))$

$$P(X<2) = \boxed{1/16}$$

(or)

(Rough)

$$P(X<2) = \int_{1}^{2} f(x)\,dx$$

$$= \int_{1}^{2} 4k(x-1)^3\,dx = 4k\int_{1}^{2}(x-1)^3\,dx$$

$$= \frac{4}{16}\left(\frac{(x-1)^4}{4}\right)_1^2$$

$$= \frac{4}{16}\cdot\frac{1}{4}\left[(2-1)^4 - (1-1)^4\right]$$

$$= \frac{4}{16}\cdot\frac{1}{4}\left[(1)^4 - 0^4\right]$$

$$= \frac{\cancel{4}}{16}\cdot\frac{1}{\cancel{4}}(1)$$

$$P(X<2) = \boxed{1/16}\ //$$

(2.40)

18) A continuous r.v X that can assume values between $x=2$ and $x=5$ has $dF = [2(1+x)/27]\,dx$ find $P(3<x<4)$

<u>Soln:</u>

W.K.T, $f(x) = \frac{dF}{dx} = \frac{2(1+x)}{27}$

(ie) $f(x) = \frac{2(1+x)}{27}, \quad 2 \leq x \leq 5$

$$P(3<X<4) = \int_3^4 f(x)\,dx$$

$$= \int_3^4 \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\int_3^4 (1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_3^4$$

$$= \frac{2}{27}\left[(4+8) - \left(3+\frac{9}{2}\right)\right]$$

$$= \frac{2}{27}\left[9 - \frac{9}{2}\right]$$

$$= \frac{2}{27}\left[\frac{9}{2}\right]$$

$$\boxed{P(3<X<4) = \frac{1}{3}}$$

Two dimensional random Variables:

Let S be the Sample Space associated with a random experiment E.

Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcomes $s \in S$. Then (X, Y) is called a two-dimensional Random variables.

Joint Probability Mass function of (X, Y).

The function $P(x, y)$ is the Joint Probability mass function of the discrete random variable (X, Y), if

(i) $P(X = x_i, Y = y_j) \geq 0$ where $i = 1, 2, \cdots, n$ & $j = 1, 2, \cdots, m$

(ii) $\sum_i \sum_j P(X = x_i, Y = y_j) = 1$

we denote $P(X = x_i, Y = y_j) = P(x_i, y_j) = P_{ij}$.

Joint Probability density function:

If (X, Y) is a two-dimensional continuous R.V such that

$$P\left\{x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2} \text{ and } y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\right\}$$
$$= f(x, y)\, dx\, dy.$$

then $f(x,y)$ is called the joint pdf of (X,Y) provided $f(x,y)$ satisfies the following conditions.

(i) $f(x,y) \geq 0$ for all $(x,y) \in R$ where R is the range space.

(ii) $\iint_R f(x,y)\,dx\,dy = 1$.

Note:

$$P[x_1 < X < x_2,\ y_1 < Y < y_2] = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx.$$

Joint cumulative distribution function:

For the random variable (X,Y) the cumulative distribution function is

$$F(x,y) = P[X \leq x,\ Y \leq y]$$

(i) Discrete case: $F(x,y) = \sum\limits_{j}\sum\limits_{i} p_{ij}$

$y_j \leq y;\ x_i \leq x$

(ii) continuous case: $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx$

Properties of cumulative distribution function:

(i) $0 \le F(x,y) \le 1$

(ii) $P[a < X < b,\ Y \le y] = F(b,y) - F(a,y)$

(iii) $P[X \le x,\ c < y < d] = F(x,d) - F(x,c)$

(iv) $P[a < X < b,\ c < Y < d] = F(b,d) - F(a,d) - F(b,c) + F(a,c)$

(v) $F(x,y)$ is a non-decreasing function.

(vi) $F(-\infty, y) = 0$; $F(x, -\infty) = 0$; $F(\infty, \infty) = 1$

(vii) At the points of continuity of $f(x,y)$,

$$\frac{\partial^2}{\partial x \cdot \partial y} F(x,y) = f(x,y)$$

where $f(x,y)$ is the joint p.d.f of (X,Y).

Marginal distributions:

Let (X,Y) be a two dimensional r.v.

Discrete Case:

The marginal distribution for X alone is given by

$$P[X = x_i] = \sum_j P(X = x_i, Y = y_j) = \boxed{p_{i.}}$$

The marginal distribution for Y alone is given by

$$P[Y = y_j] = \sum_i P(X = x_i, Y = y_j) = \boxed{p_{.j}}$$

Continuous case:

The marginal distribution for X alone is given by $\boxed{f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy}$

The marginal distribution for Y alone is given by $\boxed{f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx}$

Conditional Probability distribution:

Discrete case:

Let $P[X = x_i, Y = y_j]$ be the joint Probability function of a two dimensional R.V (X,Y) then the conditional probability function of X given $Y = y_j$ is defined by

$$P[X = x_i \mid Y = y_j] = \frac{P[X = x_i \cap Y = y_j]}{P[Y = y_j]} = \boxed{\frac{p_{ij}}{p_{\cdot j}}}$$

Similarly, the conditional Probability function of Y is given $X = x_i$ is defined by

$$P[Y = y_j \mid X = x_i] = \frac{P[X = x_i \cap Y = y_j]}{P[X = x_i]} = \boxed{\frac{p_{ij}}{p_{i\cdot}}}$$

Continuous case:

Let (X, Y) be the two dimensional continuous R.V with joint p.d.f $f(x,y)$.

Then the conditional p.d.f of X given Y is
$$\boxed{f(x/y) = \frac{f(x,y)}{f_Y(y)}}$$

where $f_Y(y)$ is the marginal p.d.f of Y.

Similarly, the conditional p.d.f of Y given X is
$$\boxed{f(y/x) = \frac{f(x,y)}{f_X(x)}}$$

where $f_X(x)$ is the marginal p.d.f of X.

Independence of two Random variables:

Discrete case:

If (X, Y) is a two-dimensional discrete random variable such that

$$P[X = x_i / Y = y_j] = P(X = x_i)$$

ie., $\dfrac{p_{ij}}{p_{.j}} = p_{i.}$ ie., $p_{ij} = p_{i.} \times p_{.j}$ for all i, j.

then X and Y are said to be Independent R.V.

Continuous Case:

If (X,Y) is a two-dimensional continuous random variable with joint p.d.f $f(x,y)$ such that $\boxed{f(x,y) = f_X(x) \cdot f_Y(y)}$. then X and Y are said to be independent random variables.

Problems:

1) From the following distribution of (X,Y), find (i) $P(X \le 1)$, (ii) $P(Y \le 3)$, (iii) $P(X \le 1, Y \le 3)$, (iv) $P(X \le 1 / Y \le 3)$, (v) $P(Y \le 3 / X \le 1)$, (vi) $P(X+Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

2.47

Soln:

$x \backslash y$	1	2	3	4	5	6	$p_{i\cdot}$ $P(X=x)$
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$	$P(X=0)=8/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$	$P(X=1)=10/16$
2	$1/38$	$1/38$	$1/64$	$1/64$	0	$2/64$	$P(X=2)=8/64$
$p_{\cdot j}$ $P(Y=y)$	$P(Y=1)=\frac{3}{32}$	$P(Y=2)=3/32$	$P(Y=3)=11/64$	$P(Y=4)=13/64$	$P(Y=5)=6/32$	$P(Y=6)=16/64$	$\sum_{ij} p_{ij}=\frac{64}{64}=1$

(i) $P(X\le 1) = P(X=0)+P(X=1)$

$$= \frac{8}{32}+\frac{10}{16}$$

$$= \frac{28}{32} = 7/8$$

$$\boxed{P(X\le 1)=7/8}$$

(ii) $P(Y\le 3) = P(Y=1)+P(Y=2)+P(Y=3)$

$$= \frac{3}{32}+\frac{3}{32}+\frac{11}{64}$$

$$\boxed{P(Y\le 3) = \frac{23}{64}}$$

(iii) $P(X\le 1, Y\le 3) = P(0,1)+P(0,2)+P(0,3)+P(1,1)+P(1,2)+P(1,3) = 0+0+\frac{1}{32}+\frac{1}{16}+\frac{1}{16}+\frac{1}{8}$

$$\boxed{P(X\le 1, Y\le 3) = \frac{9}{32}}$$

(iv) $P(X \leq 1 / Y \leq 3) = \dfrac{P(X \leq 1, Y \leq 3)}{P(Y \leq 3)}$

$= \dfrac{9/32}{23/64} = \dfrac{9}{32} \times \dfrac{64}{23}$

$\boxed{P(X \leq 1 / Y \leq 3) = \dfrac{18}{23}}$

(v) $P(Y \leq 3 | X \leq 1) = \dfrac{P(Y \leq 3, X \leq 1)}{P(X \leq 1)}$

$= \dfrac{9/32}{7/8} = \dfrac{9}{32} \times \dfrac{8}{7}$

$\boxed{P(Y \leq 3 | X \leq 1) = 9/28}$

(vi) $P(X+Y \leq 4) = P(0,1) + P(0,2) + P(0,3) + P(0,4)$
$+ P(1,1) + P(1,2) + P(1,3) + P(2,1) + P(2,2)$

$= 0 + 0 + 1/32 + 2/32 + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32}$

$\boxed{P(X+Y \leq 4) = \dfrac{13}{32}}$

2) The joint probability function (X, Y) is given by $P(x,y) = k(2x+3y)$ where $x = 0,1,2$, $y = 1,2,3$.

(i) Find the marginal distribution.
(ii) Find the ~~probability~~ distribution of $(X+Y)$
(iii) Find all conditional Probability distribution.

Soln:

(i)

X \ Y	1	2	3	$P(X=x)$
0	$3K$	$6K$	$9K$	$18K$
1	$5K$	$8K$	$11K$	$24K$
2	$7K$	$10K$	$13K$	$30K$
$P(Y=y)$	$15K$	$24K$	$33K$	$\sum_{ij} p_{ij} = 72K$

W.K.T, $\sum_i \sum_j p_{ij} = 1$

$$72K = 1$$

$$\boxed{K = 1/72}$$

(a) Marginal distribution of X-alone.

X	0	1	2
$P(X=x)$	$\frac{18}{72}$	$\frac{24}{72}$	$\frac{30}{72}$

(b) Marginal distribution of Y-alone.

Y	1	2	3
$P(Y=y)$	$15/72$	$24/72$	$33/72$

(ii) Probability distribution of $X+Y$

$X+Y$	$P(X+Y)$
$(0,1)$	$3k = 3/72$
$(0,2)$, $(1,1)$	$6k+5k = 11/72$
$(0,3)$, $(1,2)$, $(2,1)$	$9k+8k+7k = 24/72$
$(1,3)$, $(2,2)$	$11k+10k = 21/72$
$(2,3)$	$13k = 13/72$
Total	1

(iii) (1) The conditional distribution X given Y is $P[X=x_i / Y=y_j]$

(a) $$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{3/72}{15/72} = \frac{1}{5}$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{5/72}{15/72} = \frac{1}{3}$$

$$P(X=2/Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{7/72}{15/72} = \frac{7}{15}$$

(b) $P(X=0/Y=2) = \dfrac{P(X=0, Y=2)}{P(Y=2)} = \dfrac{6/72}{24/72} = \dfrac{1}{4}$

$P(X=1/Y=2) = \dfrac{P(X=1, Y=2)}{P(Y=2)} = \dfrac{8/72}{24/72} = \dfrac{1}{3}$

$P(X=2/Y=2) = \dfrac{P(X=2, Y=2)}{P(Y=2)} = \dfrac{10/72}{24/72} = \dfrac{5}{12}$

(c) $P(X=0/Y=3) = \dfrac{P(X=0, Y=3)}{P(Y=3)} = \dfrac{9/72}{33/72} = \dfrac{9}{33} = \dfrac{3}{11}$

$P(X=1/Y=3) = \dfrac{P(X=1, Y=3)}{P(Y=3)} = \dfrac{11/72}{33/72} = \dfrac{1}{3}$

$P(X=2/Y=3) = \dfrac{P(X=2, Y=3)}{P(Y=3)} = \dfrac{13/72}{33/72} = \dfrac{13}{33}$

(2) The conditional distribution Y given X is

$P[Y = y_j / X = x_i]$

(a) $P(Y=1/X=0) = \dfrac{P(Y=1, X=0)}{P(X=0)} = \dfrac{3/72}{18/72} = 1/6$

$P(Y=2/X=0) = \dfrac{P(Y=2, X=0)}{P(X=0)} = \dfrac{6/72}{18/72} = 1/3$

$P(Y=3/X=0) = \dfrac{P(Y=3, X=0)}{P(X=0)} = \dfrac{9/72}{18/72} = \dfrac{1}{2}$

(b) $P[Y=1/X=1] = \dfrac{P(Y=1, X=1)}{P(X=1)} = \dfrac{5/72}{24/72} = 5/24$

$P[Y=2/X=1] = \dfrac{P(Y=2, X=1)}{P(X=1)} = \dfrac{8/72}{24/72} = 1/3$

$P[Y=3/X=1] = \dfrac{P(Y=3, X=1)}{P(X=1)} = \dfrac{11/72}{24/72} = 11/24$

(c) $P[Y=1/X=2] = \dfrac{P(Y=1, X=2)}{P(X=2)} = \dfrac{7/72}{30/72} = 7/30$

$P[Y=2/X=2] = \dfrac{P[Y=2, X=2]}{P[X=2]} = \dfrac{10/72}{30/72} = 1/3$

$P[Y=3/X=2] = \dfrac{P[Y=3, X=2]}{P[X=2]} = \dfrac{13/72}{30/72} = \dfrac{13}{30}$

H.W

3) The two dimensional random variable (X, Y) has the joint density function $f(x,y) = \dfrac{x+2y}{27}$, $x = 0, 1, 2$; $y = 0, 1, 2$. Find the conditional distribution of Y given $X = x$. Also find the conditional distribution of X given $Y = 1$.

4) The joint probability distribution of (X, Y) where, X and Y are discrete is given in the following table. (2.53)

X \ Y	0	1	2
0	0.1	0.04	0.06
1	0.2	0.08	0.12
2	0.2	0.08	0.12

(i) Find the marginal distribution.

(ii) Verify whether X and Y are independent.

Soln:

(i)

X \ Y	0	1	2	$P_{i\cdot}$ $P(X=x)$
0	0.1	0.04	0.06	0.2
1	0.2	0.08	0.12	0.4
2	0.2	0.08	0.12	0.4
$P_{\cdot j}$ $P(Y=y)$	0.5	0.2	0.3	$\sum_{ij} P_{ij} = 1$

Marginal distribution of X :

X	0	1	2
$P(X=x)$	0.2	0.4	0.4

Marginal distribution of Y:

X	0	1	2
$P(Y=y)$	0.5	0.2	0.3

(ii) If X & Y are independent then

$$p_{ij} = p_{i.} \times p_{.j}$$

* $p_{0.} \times p_{.0} = 0.2 \times 0.5 = 0.1 = p_{00}$

$p_{0.} \times p_{.1} = 0.2 \times 0.2 = 0.04 = p_{01}$

$p_{0.} \times p_{.2} = 0.2 \times 0.3 = 0.06 = p_{02}$

* $p_{1.} \times p_{.0} = 0.4 \times 0.5 = 0.2 = p_{10}$

$p_{1.} \times p_{.1} = 0.4 \times 0.2 = 0.08 = p_{11}$

$p_{1.} \times p_{.2} = 0.4 \times 0.3 = 0.12 = p_{12}$

* $p_{2.} \times p_{.0} = 0.4 \times 0.5 = 0.2 = p_{20}$

$p_{2.} \times p_{.1} = 0.4 \times 0.2 = 0.08 = p_{21}$

$p_{2.} \times p_{.2} = 0.4 \times 0.3 = 0.12 = p_{22}$

$\therefore$ X and Y are independent.

5) Let X and Y have the joint probability mass function

Y \ X	0	1	2
0	0.1	0.4	0.1
1	0.2	0.2	0

(i) $P(X+Y>1)$

(ii) The pmf $P(X=x)$ of the R.V X

(iii) $P(Y=1/X=1)$

Soln:

(i)

X + Y	P (X + Y)
(2, 0)	0.1
(1, 1)	0.2
(2, 1)	0
Total	0.3

(ii) The pmf $P(X=x)$ of the R.V X :

X	0	1	2
$P(X=x)$	0.3	0.6	0.1

(iii) $P(Y=1/X=1) = \dfrac{P(Y=1, X=1)}{P(X=1)} = \dfrac{0.2}{0.6} = \boxed{\dfrac{1}{3}}$

6) The joint Probability density function. (2.56)
(JPdf) of the R.V (X,Y) is given by

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4} & ; \; 0<x<2, \; 0<y<1 \\ 0 & ; \; \text{otherwise} \end{cases}$$

i) Find Marginal density functions of X & Y.

(ii) Conditional density function of X given Y.

Soln:

(i) Marginal distribution of X alone:

W.K.T, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$$= \int_0^1 \frac{x(1+3y^2)}{4}\, dy$$

$$= \frac{1}{4}\int_0^1 x(1+3y^2)\, dy$$

$$= \frac{1}{4}\left[xy + \frac{3xy^3}{3}\right]_0^1$$

$$= \frac{1}{4}[x+x]$$

$$= \frac{2x}{4} = \boxed{\frac{x}{2}}$$

$$\boxed{f_X(x) = \frac{x}{2}, \quad 0<x<2}$$

Marginal distribution of y alone:

W.K.T, $f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_0^2 \left[\frac{x(1+3y^2)}{4}\right] dx$$

$$= \frac{1}{4}\int_0^2 (x+3xy^2)\,dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + \frac{3x^2y^2}{2}\right]_0^2$$

$$= \frac{1}{4}\left[\frac{4}{2} + \frac{12y^2}{2}\right]$$

$$= \frac{1}{4}[2+6y^2] = \frac{1}{2}(1+3y^2)$$

$$\boxed{f_y(y) = \frac{1}{2}(1+3y^2), \quad 0<y<1}$$

(ii) W.K.T, $f(x/y) = \dfrac{f(x,y)}{f_y(y)}$

$$= \frac{\dfrac{x(1+3y^2)}{4}}{\dfrac{1}{2}(1+3y^2)}$$

$$\boxed{f(x/y) = \frac{x}{2}, \quad 0<x<2, \quad 0<y<1}$$

7) If the joint pdf of (X,Y) is given by $f(x,y)=2, \quad 0\le x\le y\le 1$.

Find (i) Marginal density function of x & y

(ii) Conditional densities $f(x/y)$ & $f(y/x)$.

(i) Marginal density function of x:

W.K.T, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_{x}^{1} 2\,dy$

$= [2y]_x^1$

$= 2(1-x), \quad 0 \le x \le 1.$

$$\boxed{f_X(x) = 2(1-x), \quad 0 \le x \le 1}$$

Marginal density function of y:

W.K.T, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \int_{0}^{y} 2\,dx$

$= 2[x]_0^y$

$$\boxed{f_Y(y) = 2y, \quad 0 \le y \le 1}$$

(ii) Conditional density function of $f(x/y)$:

$f(x|y) = \dfrac{f(x,y)}{f(y)}$

$= \dfrac{2}{2y}$

$$\boxed{f(x|y) = \frac{1}{y}, \quad 0 \le x \le y \le 1}$$

Conditional density function of $f(y/x)$:

$$f(y/x) = \frac{f(y,x)}{f(x)}$$

$$= \frac{2}{2(1-x)}$$

$$\boxed{f(y/x) = \frac{1}{1-x}, \quad 0 \le x \le y \le 1}$$

8) The joint p.d.f of a RV (X,Y) is given by

$$f(x,y) = kxy\,e^{-(x^2+y^2)}, \quad x>0, y>0$$

(i) find k

(ii) prove that X & Y are independent.

Soln:

(i) To find k:

W.K.T, $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^{\infty}\int_0^{\infty} kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1$$

$$\Rightarrow k\int_0^{\infty} x e^{-x^2}\,dx \int_0^{\infty} y e^{-y^2}\,dy = 1$$

$x^2 = t$	$y^2 = s$
$2x\,dx = dt$	$2y\,dy = ds$
$x\,dx = dt/2$	$y\,dy = \frac{ds}{2}$
$x=0 \Rightarrow t=0$	$y=0 \Rightarrow s=0$
$x=\infty \Rightarrow t=\infty$	$y=\infty \Rightarrow s=\infty$

$$\Rightarrow K\int_0^\infty e^{-t}\frac{dt}{2}\int_0^\infty e^{-s}\frac{ds}{2}=1$$

$$\Rightarrow \frac{K}{4}\left(\frac{e^{-t}}{-1}\right)_0^\infty\left(\frac{e^{-s}}{-1}\right)_0^\infty=1$$

$$\Rightarrow \frac{K}{4}\left[(e^{-\infty}-e^{0})(e^{-\infty}-e^{0})\right]=1$$

$$\Rightarrow \frac{K}{4}\left[(-1)(-1)\right]=1$$

$$\Rightarrow \boxed{K=4}$$

(ii) To prove:

$$f(x,y)=f_X(x),\ f_Y(y)$$

W.K.T, $f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$

$$=\int_0^\infty Kxy\,e^{-(x^2+y^2)}\,dy$$

$$=K\int_0^\infty xy\,e^{-(x^2+y^2)}\,dy$$

$$=Kxe^{-x^2}\int_0^\infty ye^{-y^2}\,dy$$

$$=Kxe^{-x^2}\int_0^\infty e^{-s}\frac{ds}{2}$$

$$=\frac{4}{2}xe^{-x^2}\left[\frac{e^{-s}}{-1}\right]_0^\infty$$

$$=2xe^{-x^2}(-e^{-\infty}+e^{0})$$

$$\boxed{f_X(x) = 2xe^{-x^2}, \quad 0 < x < \infty}$$

W.K.T, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

$$= \int_0^{\infty} kxy\, e^{-(x^2+y^2)} dx$$

$$= ky e^{-y^2} \int_0^{\infty} x e^{-x^2} dx$$

$$= ky e^{-y^2} \int_0^{\infty} e^{-t} \frac{dt}{2}$$

$$= \frac{4}{2} y e^{-y^2} \left[\frac{e^{-t}}{-1} \right]_0^{\infty}$$

$$= -2ye^{-y^2} \left[e^{-\infty} - e^0 \right]$$

$$f_Y(y) = 2ye^{-y^2}, \quad 0 < y < \infty$$

$$\therefore f(x,y) = f_X(x) \cdot f_Y(y)$$

$$= 2xe^{-x^2} \cdot 2ye^{-y^2}$$

$$= 4xye^{-(x^2+y^2)}, \quad 0 < x < \infty; \; 0 < y < \infty$$

$\therefore f(x,y) = f_X(x) \cdot f_Y(y)$

$\therefore$ X & Y are independent.

9) The joint pdf of (X, Y) is given by

$$f(x,y) = \begin{cases} \dfrac{x^3 y^3}{16}, & 0 \le x \le 2, \quad 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the marginal densities of X & Y

(ii) Find the cumulative distr. fn. of x & y

(iii) P.T x & y are independent.

Soln:

(i) Marginal density function of X:

W.K.T, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$$f_X(x) = \int_0^2 \frac{x^3 y^3}{16}\, dy$$

$$= \frac{1}{16} \int_0^2 x^3 y^3\, dy$$

$$= \frac{x^3}{16} \left[\frac{y^4}{4} \right]_0^2$$

$$= \frac{x^3}{16} \left[\frac{16}{4} - 0 \right]$$

$$= \frac{x^3}{16} \cdot \frac{16}{4}$$

$$\boxed{f_X(x) = \frac{x^3}{4}, \quad 0 \le x \le 2}$$

Marginal density function of Y:

W.K.T, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

$$= \int_0^2 \frac{x^3 y^3}{16}\, dx$$

$$= \frac{y^3}{16}\left[\frac{x^4}{4}\right]_0^2$$

$$= \frac{y^3}{16} \cdot \frac{16}{4}$$

$$\boxed{f_y(y) = \frac{y^3}{4}, \quad 0 \le y \le 2}$$

(iii) To prove: X & Y are independent!

w.k.T, $f(x,y) = f_x(x) \cdot f_y(y)$

$$= \frac{x^3}{4} \cdot \frac{y^3}{4}$$

$$f(x,y) = \frac{x^3 \cdot y^3}{16}, \quad 0 \le x \le 2, \quad 0 \le y \le 2$$

$$\therefore \boxed{f(x,y) = f_x(x) \cdot f_y(y)}$$

$\therefore$ X and Y are independent.

(ii) $F_x(x) = \int_0^x f_x(x)\, dx = \int_0^x \frac{x^3}{4}\, dx$

$$= \frac{1}{4}\left(\frac{x^4}{4}\right)_0^x$$

$$F_x(x) = \frac{x^4}{16}, \quad 0 \le x \le 2$$

$$\boxed{F_X(x) = \begin{cases} 0 & , \ x < 0 \\ \dfrac{x^4}{16} & , \ 0 \le x \le 2 \\ 1 & , \ x > 2 \end{cases}}$$

$$F_Y(y) = \int_0^y f_Y(y)\,dy$$

$$= \int_0^y \frac{y^3}{4}\,dy$$

$$= \frac{1}{4}\left(\frac{y^4}{4}\right)_0^y = \frac{y^4}{16}, \quad 0 \le y \le 2$$

$$\boxed{F_Y(y) = \begin{cases} 0 & , \ y < 0 \\ \dfrac{y^4}{16} & , \ 0 < y < 2 \\ 1 & , \ y > 2 \end{cases}}$$

H.W.

10) Given the joint pdf of X & Y is

$$f(x,y) = \begin{cases} 8xy & ; \ 0 < x < y < 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Find the marginal & conditional p.d.f of X & Y. Are X & Y independent?

Ans: $\left.\begin{array}{l} f_X(x) = 4x(1-x^2), \ 0 < x < 1 \quad \& \\ f_Y(y) = 4y^3, \ 0 < y < 1 \end{array}\right\}$ Marginal

conditional $\boxed{f(X/Y) = 2x/y^2, \ 0 < x < y < 1 \ \ \& \ \ f(Y/X) = \frac{2y}{1-x^2}, \ 0 < x < y < 1}$

X & Y are not independent.

Mathematical Expectation and Generating Functions:

Expectation:

The averaging process, when applied to a random variable is called expectation. It is denoted by $E(X)$ and is read as the expected value of X or the mean value of X.

Case (i)

If X is discrete R.V then

$$E(X) = \sum_i x_i \, p(x_i)$$

Case (ii)

If X is a continuous R.V then

$$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

Case (iii)

Mathematical expectation of some real function $g(x)$ is given by

For discrete case,

$$E[g(x)] = \sum_i g(x_i) \, p(x_i)$$

For continuous case,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx$$

Properties of Mathematical Expectation:

If X and Y are R.V and a, b are constants then:

(i) $E(a) = a$

(ii) $E(aX) = a\,E(X)$

(iii) $E(aX+b) = E(aX) + E(b)$

(iv) Expectation of additive property,

$$E(X+Y) = E(X) + E(Y)$$

(v) Expectation of multiplicative property,

$E(XY) = E(X) \cdot E(Y)$ [$\because$ X & Y are independent R.V]

(vi) $E(X-\overline{X}) = 0$

(vii) $E(X) \geq 0$, if $X \geq 0$

(viii) $|E(X)| \leq E(|X|)$

Note:

(i) Mean of $X = \overline{X} = E(X)$

(ii) Variance of X $[\mathrm{Var}(X)] = \sigma_X^2$

$$= E(X^2) - [E(X)]^2$$

(or)

$$= E[(x-\overline{x})^2]$$

where $\overline{X}$ is the mean value (or) expected value of the R.V X.

Properties of Variance:

(i) $var(X) \geq 0$

(ii) $Var(a) = 0$, a is any constant.

(iii) If X is a R.V, then $\boxed{V(aX) = a^2 V(X)}$, where a is constant.

Proof:

Let $Y = aX$

$$E(Y) = E(aX)$$
$$= a\,E(X)$$
$$\therefore E(Y^2) = E(a^2X^2)$$
$$= a^2 E(X^2)$$
$$\therefore var(Y) = E(Y^2) - [E(Y)]^2$$
$$= a^2 E(X^2) - [a\,E(X)]^2$$
$$= a^2 E(X^2) - a^2 [E(X)]^2$$
$$= a^2 \left[E(X^2) - [E(X)]^2 \right]$$
$$= a^2 \, var(X)$$
$$\therefore \boxed{Var(aX) = a^2 \, var(X)}$$

(iv) $var(aX \pm bY) = a^2 V(X) + b^2 V(Y)$ (if X & Y are independent)

Moments:

Type: (1) Moments about the origin (Raw moments).

Case (i) (For discrete r.v)

For discrete r.v X, the r^{th} raw moment is

$$\boxed{E(X^r) = \sum_i x_i^r p_i = \mu_r' \quad , \quad r \geq 1}$$

Case (ii). (For continuous r.v)

For a continuous r.v X, the r^{th} raw moment is

$$\boxed{E(X^r) = \int_{-\infty}^{\infty} x^r f_X(x)\, dx = \mu_r' \quad , \quad r \geq 1}$$

Type: (2). Moments about the mean (central moments)

Case (i). (For discrete r.v X)

The r^{th} moment about the mean of a discrete r.v X is

$$\boxed{E\left[(X-\bar{X})^r\right] = \sum_i (x_i - \bar{X})^r p_i = \mu_r}$$

Case (ii). (For continuous r.v X)

The r^{th} moment about the mean of a continuous r.v X is

$$\boxed{E\left[(X-\bar{X})^r\right] = \int_{-\infty}^{\infty} (x-\bar{X})^r f_X(x)\, dx = \mu_r}$$

Relation between raw moments and central moments:

$$\mu_1 = 0 \text{ (always)}$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = E(x^2) - [E(x)]^2 = \text{var}(x)$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

Problems:

1) If X is a R.V having density function

$$f(x) = \begin{cases} x/6, & x = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

Find mean, variance and standard deviation.

Also find $E[4x^3 + 3x + 11]$

Soln:

This is a discrete R.V.

The Probability distribution is

X	1	2	3
$P(X=x)$	$1/6$	$2/6$	$3/6$

(i) Mean:

$$E(X) = \sum_i x_i p_i$$

$$= x_1 p_1 + x_2 p_2 + x_3 p_3$$

$$= 1(1/6) + 2(2/6) + 3(3/6)$$

$$= \frac{1}{6} + \frac{4}{6} + \frac{9}{6}$$

$$= \boxed{\frac{14}{6}}$$

$$E(X) = \frac{7}{3} = 2.33$$

(ii) Variance:

$$E(x^2) = \sum_i x_i^2 p_i$$

$$= x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3$$

$$= 1^2(1/6) + 2^2(2/6) + 3^2(3/6)$$

$$= 1/6 + 8/6 + 27/6$$

$$= 36/6$$

$$\boxed{E(x^2) = 6}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= 6 - (7/3)^2$$

$$= 6 - \frac{49}{9}$$

$$= \frac{54 - 49}{9}$$

$$\boxed{Var(x) = \frac{5}{9} = 0.5571}$$

(iii) Standard Deviation:

$$S.D = \sqrt{Var(x)}$$

$$= \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3} = \frac{2.236}{3} = \boxed{0.745}$$

(iv) $E(4X^3+3X+11) = E(4X^3)+E(3X)+E(11)$

$= 4E(X^3)+3E(X)+11 \longrightarrow (1)$

$E(X^3) = \sum_i x_i^3 p_i$

$= x_1^3 p_1 + x_2^3 p_2 + x_3^3 p_3$

$= 1^3(1/6) + 2^3(2/6) + 3^3(3/6)$

$= \frac{1}{6} + \frac{16}{6} + \frac{81}{6}$

$= \frac{98}{6}$

$\boxed{E(X^3) = 16.333}$

$(1) \Rightarrow E(4X^3+3X+11) = 4(16.333) + 3(2.33) + 11$

$\boxed{E(4X^3+3X+11) = 83.331}$

2) When a die is thrown, X denotes the number that turns up. Find $E(X)$, $E(X^2)$, $Var(X)$ and S.D.

Soln:

Probability distribution:

X	1	2	3	4	5	6
P(X=x)	1/6	1/6	1/6	1/6	1/6	1/6

(i) $E(X) = \sum_i x_i p_i$

$= 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6)$

$$= \frac{1+2+3+4+5+6}{6}$$

$$E(X) = \frac{21}{6} = \boxed{3.5}$$

(ii) $E(X^2) = \sum_i x_i^2 p_i$

$$= 1^2(1/6) + 2^2(1/6) + 3^2(1/6) + 4^2(1/6) + 5^2(1/6) + 6^2(1/6)$$

$$= \frac{1+4+9+16+25+36}{6}$$

$$\boxed{E(X^2) = \frac{91}{6} = 15.1666}$$

(iii) $Var(X) = E(X^2) - [E(X)]^2$

$$= \frac{91}{6} - \left(\frac{21}{6}\right)^2$$

$$= \frac{91}{6} - \frac{441}{36}$$

$$= \frac{546-441}{36}$$

$$\boxed{Var(X) = \frac{105}{36} = 2.9166}$$

(iv) $SD = \sqrt{Var(X)}$

$$= \sqrt{2.9166}$$

$$\boxed{S.D = 1.7078}$$

3) The R.V X can only take the value 2 & 5. (3.9)
Given that the value 5 is twice as likely the value 2, determine the expectation of X.

Soln: Given $x = 2, 5$

Let $P(X=2) = p$ &

$P(X=5) = 2p$

The Probability distribution is

X	2	5
$P(X=x)$	p	$2p$

Here, $p + 2p = 1$

$\Rightarrow 3p = 1$

$\Rightarrow \boxed{p = 1/3}$

Now, the probability distribution is

X	2	5
$P(X=x)$	$1/3$	$2/3$

$\therefore E(X) = \sum_i x_i p_i$

$= 2(1/3) + 5(2/3)$

$= \frac{2}{3} + \frac{10}{3}$

$= 12/3 = \boxed{4}$

4) A coin is tossed until a head appears. What is the expectation of the no. of tosses required? (3.10)

Soln: Let X denote the number of tosses required to get the first head.

The Probability distribution of X:

X	1	2	3	4	......
$P(X=x)$	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$	....

$\left(\because \frac{P(A)}{P(S)}\right)$

$$E(X) = \sum_i x_i p_i$$

$$= 1\left(1/2\right) + 2\left(1/2^2\right) + 3\left(1/2^3\right) + \cdots$$

$$= \frac{1}{2}\left[1 + 2\left(1/2\right) + 3\left(1/2^2\right) + \cdots\right]$$

$$= \frac{1}{2}\left[1 + 2\left(1/2\right) + 3\left(1/2\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left(1 - 1/2\right)^{-2} \qquad \left(\because (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots\right)$$

$$= \frac{1}{2}\left(1/2\right)^{-2}$$

$$= \frac{1}{2}(2)^2$$

$$\boxed{\therefore E(X) = 2}$$

5) Find the expectation of the no. of failures preceding the first success in an infinite series of independent trials with constant probability of success in each trial. (3.11)

Soln:

Let X denotes the no. of failures preceding the first success.

The probability distribution of X is

X	0	1	2	3	4	....
$P(X=x)$	p	qp	q^2p	q^3p	q^4p	....

$$E(X) = \sum_i x_i p_i$$

$$= 0(p) + 1(qp) + 2(q^2p) + 3(q^3p) + 4(q^4p) + \cdots$$

$$= qp + 2q^2p + 3q^3p + 4q^4p + \cdots$$

$$= qp(1 + 2q + 3q^2 + 4q^3 + \cdots)$$

$$= qp(1-q)^{-2} \qquad (\because (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots)$$

$$= qp(p)^{-2} \qquad (\because 1-q = p \text{ (or) } p+q=1)$$

$$= qp(1/p^2)$$

$$= \frac{qp}{p^2} = \frac{q}{p}$$

$$\boxed{E(X) = \frac{q}{p}}$$

6) If X has the distribution function (3.12)

$$F(X) = \begin{cases} 0 & ; \quad x < 1 \\ 1/3 & ; \quad 1 \le x < 4 \\ 1/2 & ; \quad 4 \le x < 6 \\ 5/6 & ; \quad 6 \le x < 10 \\ 1 & ; \quad x \geq 10 \end{cases}$$

Find (i) The probability distribution of X

(ii) $P(2 < X < 6)$

(iii) Mean (X)

(iv) Var (X)

Soln:

(i)

$F(0) = 0 \Rightarrow p(0) = 0$

$F(1) = 1/3 \Rightarrow p(1) = \boxed{1/3} = F(1) - F(0) \Rightarrow 1/3 - 0 = \boxed{1/3}$

$F(4) = 1/2 \Rightarrow P(4) = F(4) - F(1) \Rightarrow 1/2 - 1/3 = \boxed{1/6}$

$F(6) = 5/6 \Rightarrow P(6) = F(6) - F(4) \Rightarrow 5/6 - 1/2 = \frac{2}{6} = \boxed{\frac{1}{3}}$

$F(10) = 1 \Rightarrow P(10) = F(10) - F(6) \Rightarrow 1 - 5/6 = \boxed{1/6}$

The probability distribution of X is

X	0	1	4	6	10
$P(X=x)$	0	1/3	1/6	1/3	1/6

(ii) $P(2 < X < 6) = P(X=4)$

$$= \boxed{1/6}$$

(iii) $E(X) = \sum_i x_i p_i$

$$= 0(0) + 1(1/3) + 4(1/6) + 6(1/3) + 10(1/6)$$

$$= 1/3 + \cancel{4}^{2}/\cancel{6}_{3} + 6/3 + \cancel{10}^{5}/\cancel{6}_{3}$$

$$= \frac{1+2+6+5}{3}$$

$$\boxed{E(X) = \frac{14}{3}}$$

$E(X^2) = \sum_i x_i^2 p_i$

$$= 0^2(0) + 1^2(1/3) + 4^2(1/6) + 6^2(1/3) + 10^2(1/6)$$

$$= \frac{1}{3} + \frac{\cancel{16}^{8}}{\cancel{6}_{3}} + \frac{36}{3} + \frac{\cancel{100}^{50}}{\cancel{6}_{3}}$$

$$= \frac{1+8+36+50}{3}$$

$$\boxed{E(X^2) = \frac{95}{3}}$$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$$= \frac{95}{3} - \left(\frac{14}{3}\right)^2$$

$$= \frac{95}{3} - \frac{196}{9} = \frac{285-196}{9}$$

$$\boxed{\mathrm{Var}(X) = 89/9}$$

7) If variance of $V(X)=4$, find $var(4x+5)$, where X is a R.V.

Soln:

By property,

$$var(aX+b) = a^2 var(X)$$

$$var(4x+5) = 4^2\, var(X) + 0 \qquad (\because var(const)=0)$$

$$= 16(4) \qquad (\because V(X)=4 \text{ (Given)})$$

$$\boxed{var(4x+5) = 64}$$

8) If X and Y are independent R.V with variance 2 and 3. Find the $var(3X+4Y)$.

Soln:

Given $var(X)=2$, $var(Y)=3$

$a=3$, $b=4$

By property,

$$var(aX+bY) = a^2 var(X) + b^2 var(Y)$$

$$var(3X+4Y) = 3^2 var(X) + 4^2 var(Y)$$

$$= 9(2) + 16(3)$$

$$= 18 + 48$$

$$\boxed{var(3X+4Y) = 66}$$ //

9) Let X be a RV with $E(X)=1$, $E[X(X-1)]=4$. (3.15)

Find $var(X)$, $var(2-3X)$ and $var(X/2)$.

Soln:

(i) Given $\boxed{E(X)=1}$ and $E[X(X-1)] = 4$

$$E[X^2-X] = 4$$

$$E[X^2]-E(X)=4$$

$$E(X^2) = 4+E(X)$$

$$E(X^2) = 4+1$$

$$\boxed{E(X^2) = 5}$$

W.K.T, $var(X) = E(X^2)-[E(X)]^2$

$$= 5-1$$

$$var(X) = \boxed{4}$$ //

(ii) var(2-3X):

By property, $var(a-bX) = var(a)+var(-bX)$

$$= 0+(-b)^2 var(X)$$

$$= b^2 var(X)$$

$$var(2-3X) = 3^2(4) \quad (\because b=3)$$

$$= 9(4)$$

$$= \boxed{36}$$ //

(iii) $var(X/2) = (1/2)^2 var(X)$

$$= \frac{1}{4}\times 4$$

$$= \boxed{1}$$ //

$(\because var(aX) = a^2 var(X))$

$(a = 1/2)$

10) A continuous R.V X has pdf (3.16)

$f(x) = kx^2 e^{-x}$, $x \geq 0$. Find k, r^{th} raw moment, mean, variance.

Soln:

Given, $f(x) = kx^2 e^{-x}$, $x \geq 0$

W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow k \int_0^{\infty} x^2 e^{-x} = 1$

By Bernoulli's formula,

$\int u\,dv = uv - u'v_1 + u''v_2 - \dots$

$u = x^2$ $(+)$ $dv = e^{-x}$

$u' = 2x$ $(-)$ $v = -e^{-x}$

$u'' = 2$ $(+)$ $v_1 = e^{-x}$

$v_2 = -e^{-x}$

$\Rightarrow k \left[x^2(-e^{-x}) - 2x(e^{-x}) + 2e^{-x} \right]_0^{\infty} = 1$

$\Rightarrow k \left[-\cancel{x^2 e^{-x}}^{\to 0} - \cancel{2x e^{-x}}^{\to 0} - 2e^{-x} \right]_0^{\infty} = 1$

$\Rightarrow k \left[0 + 2e^0 \right] = 1$

$\Rightarrow 2k = 1$

$\Rightarrow \boxed{k = 1/2}$

$\boxed{f(x) = 1/2\, x^2 e^{-x}, \; x \geq 0}$

(ii) To find the r^{th} raw moment :

i.e., To find μ_r' :

W.K.T, $\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\, dx = E(X^r)$

$$= \int_0^{\infty} x^r \cdot \frac{1}{2} x^2 e^{-x}\, dx$$

$$= \frac{1}{2} \int_0^{\infty} x^{r+2} e^{-x}\, dx$$

$$= \frac{1}{2} \int_0^{\infty} x^{(r+3)-1} e^{-x}\, dx \quad \left(\because r+2 = (r+3)-1,\ n = r+3 \right)$$

$$= \frac{1}{2} \Gamma(r+3) = \frac{1}{2} \Gamma[(r+2)+1]$$

$$\boxed{\mu_r' = \frac{1}{2}(r+2)!}$$

Gamma functions

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

$$\Gamma(n+1) = n!$$

(iii) Mean :

$\mu_r' = \frac{1}{2}(r+2)!$

Put $r = 1$,

$$\text{Mean} = \mu_1' = \frac{1}{2}(1+2)!$$

$$= \frac{1}{2}(3)!$$

$$= \frac{1}{2} 3! = \frac{1}{2}(6) = \frac{6}{2} = 3$$

$$\boxed{\text{Mean} = 3}$$

Variance:

(iv) Put $r = 2$,

$$\mu_2' = \frac{1}{2}(2+2)!$$

$$= \frac{1}{2}(4!)$$

$$= \frac{1}{2}(4 \times 3 \times 2)$$

$$\boxed{\mu_2' = 12}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= 12 - 3^2$$

$$= 12 - 9$$

$$\boxed{\text{Var} = 3}$$

11) Find the mean, variance and S.D of a continuous R.V X, if it has the density function

$$f(x) = \begin{cases} 2(x-1) ; & 1 < x < 2 \\ 0 ; & \text{otherwise} \end{cases}$$

Soln:

Mean:

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_1^2 x\,[2(x-1)]\, dx$$

$$= \int_1^2 x(2x-2)\, dx$$

$$= \int_1^2 (2x^2 - 2x)\,dx$$

$$= 2\int_1^2 (x^2 - x)\,dx$$

$$= 2\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= 2\left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= 2\left[\left(\frac{16-12}{6}\right) - \left(\frac{2-3}{6}\right)\right]$$

$$= 2\left[\frac{4}{6} - \left(-\frac{1}{6}\right)\right]$$

$$= 2\left[\frac{5}{6}\right]$$

$$= 5/3$$

$$\boxed{E(X) = 1.66}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_1^2 x^2 \left[2(x-1)\right] dx$$

$$= \int_1^2 (2x^3 - 2x^2)\,dx$$

$$= \left[\frac{2x^4}{4} - \frac{2x^3}{3}\right]_1^2$$

$$= \left[\left(\frac{16}{2} - \frac{16}{3}\right) - \left(\frac{1}{2} - \frac{2}{3}\right)\right]$$

$$= \left[\left(\frac{48-32}{6}\right) - \left(\frac{3-4}{6}\right)\right]$$

$$= \left[\frac{16}{6} - \left(-\frac{1}{6}\right)\right]$$

$$= \frac{17}{6}$$

$$\boxed{E(x^2) = 2.833}$$

Variance:

$$\sigma_x^2 = Var(X) = E(x^2) - [E(x)]^2$$

$$= \frac{17}{6} - \left(\frac{5}{3}\right)^2$$

$$= \frac{17}{6} - \frac{25}{9}$$

$$= \frac{51-50}{18}$$

$$= \frac{1}{18}$$

$$\boxed{\sigma_x^2 = 0.0558}$$

S.D:

$$S.D = \sigma_x = \sqrt{Var(x)}$$

$$= \sqrt{0.0558}$$

$$\boxed{S.D = 0.2362}$$

12) Find the mean, Variance of the R.V X which has the following density function

$$f(x) = \begin{cases} x & , \ 0 < x < 1 \\ 2-x & , \ 1 < x < 2 \\ 0 & , \ \text{otherwise} \end{cases}$$

Soln:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \left[\left(\frac{8}{2} - \frac{8}{3}\right) - \left(\frac{2}{2} - \frac{1}{3}\right)\right]$$

$$= \frac{1}{3} + \left[\left(\frac{24-16}{6}\right) - \left(\frac{6-2}{6}\right)\right]$$

$$= \frac{1}{3} + \left[\frac{8}{6} - \frac{4}{6}\right]$$

$$= \frac{1}{3} + \frac{4}{6} = \frac{2+4}{6} = \frac{6}{6} = 1$$

$$\boxed{\text{Mean} = E(X) = 1}$$

$$E(X^2) = \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx$$

$$= \left(\frac{x^4}{4}\right)_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{4} + \left[\left(\frac{64-48}{12}\right) - \left(\frac{8-3}{12}\right)\right]$$

$$= \frac{1}{4} + \left[\frac{16}{12} - \frac{5}{12}\right]$$

$$= \frac{1}{4} + \frac{11}{12} = \frac{3+11}{12} = \frac{14}{12} = \frac{7}{6}$$

$$\boxed{E(X^2) = 7/6}$$

$$var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{7}{6} - 1$$

$$\boxed{var(X) = \frac{1}{6}}$$

13) The density function of R.V X is given by

$$f(x) = \begin{cases} kx(2-x)^2, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find k, mean and variance

Soln:

W.K.T, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^2 k\left[x(2-x)^2\right] dx = 1$$

$$k \int_0^2 x\left[(4 + x^2 - 4x)\right] dx = 1$$

3.23

$$k \int_0^2 (4x + x^3 - 4x^2)\, dx = 1$$

$$k \left[\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3} \right]_0^2 = 1$$

$$k \left[\left(8 + \frac{16}{4} - \frac{32}{3}\right) - 0 \right] = 1$$

$$k \left[\frac{24 + 12 - 32}{3} \right] = 1$$

$$k \left[\frac{4}{3} \right] = 1$$

$$\boxed{k = \frac{3}{4}}$$

$$\boxed{k = 0.7500}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$$

$$= \int_0^2 x\ kx(2-x)^2\, dx$$

$$= \frac{3}{4} \int_0^2 x^2 (2-x)^2\, dx$$

$$= \frac{3}{4} \int_0^2 x^2 (4 + x^2 - 4x)\, dx$$

$$= \frac{3}{4} \int_0^2 (4x^2 + x^4 - 4x^3)\, dx$$

$$= \frac{3}{4} \left[\frac{4x^3}{3} + \frac{x^5}{5} - \frac{4x^4}{4} \right]_0^2$$

$$= \frac{3}{4}\left[\left(\frac{32}{3} + \frac{32}{5} - \frac{64}{4}\right) - 0\right]$$

$$= \frac{3}{4}\left[\frac{160 + 96 - 240}{15}\right]$$

$$= \frac{3}{4}\left[\frac{256 - 240}{15}\right]$$

$$= \frac{3}{4}\left(\frac{16}{15}\right)$$

$$\boxed{E(X) = \frac{4}{5}}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_0^2 x^2 kx(2-x)^2\, dx$$

$$= \frac{3}{4}\int_0^2 x^3(4 + x^2 - 4x)\, dx$$

$$= \frac{3}{4}\int_0^2 (4x^3 + x^5 - 4x^4)\, dx$$

$$= \frac{3}{4}\left[\frac{4x^4}{4} + \frac{x^6}{6} - \frac{4x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[16 + \frac{64}{6} - \frac{128}{5}\right]$$

$$= \frac{3}{4}\left[\frac{240 + 160 - 384}{15}\right]$$

$$= \frac{3}{4}\left[\frac{16}{15}\right] \implies \boxed{E(X^2) = \frac{4}{5}}$$

$$\text{variance} = E(X^2) - [E(X)]^2$$

$$= \frac{4}{5} - \left(\frac{4}{5}\right)^2$$

$$= \frac{4}{5} - \frac{16}{25}$$

$$= \frac{100-80}{125} = \frac{20}{125}$$

$$\boxed{\text{var}(X) = \frac{4}{25}}$$

14) The cumulative distribution function of a r.v X is $F(x) = 1-(1+x)e^{-x}$, $x>0$. Find the probability density function of X. Also find the mean and variance of X.

Soln:

Given c.d.f $= F(x) = 1-(1+x)e^{-x}$, $x>0$

$$\text{p.d.f} = f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}\left[1-e^{-x}-xe^{-x}\right]$$

$$= 0 + \cancel{e^{-x}} + xe^{-x} - \cancel{e^{-x}}$$

$$= xe^{-x}, \quad x>0$$

$$\boxed{f(x) = xe^{-x}, \quad x>0}$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^{\infty} x \cdot xe^{-x}\, dx$$

$$= \int_0^{\infty} x^2 e^{-x}\, dx$$

$$u = x^2 \xleftrightarrow{(+)} dv = e^{-x}\,dx$$
$$u' = 2x \xleftrightarrow{(-)} v = -e^{-x}$$
$$u'' = 2 \xleftrightarrow{(+)} v_1 = e^{-x}$$
$$v_2 = -e^{-x}$$

$$E(X) = \left[-\cancel{x^2 e^{-x}}^{\to 0} - \cancel{2x e^{-x}}^{\to 0} - 2e^{-x}\right]_0^\infty$$

$$= \left[-2e^{-\infty} + 2e^0\right] = 0 + 2 = 2$$

$$\boxed{E(X) = 2} \Rightarrow \boxed{\text{Mean} = 2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^\infty x^2\, x e^{-x}\,dx = \int_0^\infty x^3 e^{-x}\,dx$$

$$u = x^3 \xleftrightarrow{(+)} dv = e^{-x}\,dx$$
$$u' = 3x^2 \xleftrightarrow{(-)} v = -e^{-x}$$
$$u'' = 6x \xleftrightarrow{(+)} v_1 = e^{-x}$$
$$u''' = 6 \xleftrightarrow{(-)} v_2 = -e^{-x}$$
$$v_3 = e^{-x}$$

$$E(X^2) = \left[-\underbrace{x^3 e^{-x}}_{0} - \underbrace{3x^2 e^{-x}}_{0} - \underbrace{6x e^{-x}}_{0} - 6e^{-x}\right]_0^\infty$$

$$= \left[-6e^{-\infty} + 6e^0\right] = 0 + 6 = 6$$

$$\boxed{E(X^2) = 6}$$

Variance:

$$\text{var}(X) = E(X^2) - [E(X)]^2$$
$$= 6 - (2)^2 = 6 - 4 = 2$$

$$\boxed{\text{var}(X) = \sigma_X^2 = 2}$$

Moment Generating Function (MGF)

Definition: (MGF)

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum e^{tx} p(x) & [\text{for Discrete R.V}] \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx & [\text{for Continuous R.V}] \end{cases}$$

Derivation:

$$\because e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$M_X(t) = E\left[e^{tX}\right]$$

$$= E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \dots\right]$$

$$= E(1) + \frac{t}{1!} E(X) + \frac{t^2}{2!} E(X^2) + \dots$$

$$M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

$$\therefore \boxed{\mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)}$$

$$E(X^r) = \mu_r' = \begin{cases} \sum x^r p(x), & \text{if } X \text{ is discrete R.V} \\ \int_{-\infty}^{\infty} x^r f_X(x)\, dx, & \text{if } X \text{ is continuous R.V} \end{cases}$$

Since $M_X(t)$ generates moments, hence it is known as moment generating function.

Results:

(i) $\mu_r' = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}$

(ii) $M_{cX}(t) = M_X(ct)$, c being a constant.

Proof:

$$L.H.S = M_{cX}(t)$$
$$= E\left[e^{tcX}\right] \rightarrow (1)$$

$$R.H.S = M_X(ct)$$
$$= E\left[e^{ctX}\right] \rightarrow (2)$$

$$(1) = (2)$$

$$\therefore L.H.S = R.H.S$$

(iii) The moment generating function of the sum of a given number of independent R.V is equal to the product of their respective moment generating functions.

i.e., $M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$

By definition,

$$M_X(t) = E\left(e^{tX}\right)$$

$$M_{X_1+X_2+\cdots+X_n}(t) = E\left[e^{t(X_1+X_2+\cdots+X_n)}\right]$$
$$= E\left[e^{tX_1+tX_2+\cdots+tX_n}\right]$$

$$= E\left[e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}\right]$$

$$= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \cdots E\left[e^{tX_n}\right]$$

$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$

($\because X_1, X_2, \ldots, X_n$ are independent R.V)

Hence Proved.

Problems:

1) If X is a discrete R.V with probability function $p(x) = \frac{1}{k^x}$, $x = 1, 2, \ldots$ (k is const.).

Find the (i) MGF (ii) Mean (iii) Variance.

Soln:

(i) MGF:

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{k^x}$$

$$= e^{t} \cdot \frac{1}{k} + e^{2t} \cdot \frac{1}{k^2} + e^{3t} \frac{1}{k^3} + \cdots$$

$$= \frac{e^t}{k}\left[1 + \frac{e^t}{k} + \frac{e^{2t}}{k^2} + \cdots\right]$$

$$= \frac{e^t}{k}\left[1+\frac{e^t}{k}+\left(\frac{e^t}{k}\right)^2+\cdots\right]$$

$$= \frac{e^t}{k}\left[1-\frac{e^t}{k}\right]^{-1}$$

$$= \frac{e^t}{k}\left[\frac{k-e^t}{k}\right]^{-1}$$

$$= \frac{e^t}{k}\left[\frac{k}{k-e^t}\right]$$

$$= \frac{e^t}{k-e^t}$$

$$M_x(t) = \frac{e^t}{k-e^t}$$

(ii) **Mean:**

W.K.T, $\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$

Put $r=1$,

$$\mu_r' = \left(\frac{d}{dt}\left[\frac{e^t}{k-e^t}\right]\right)_{t=0}$$

$$= \left[\frac{(k-e^t)e^t - e^t(\cancel{-e^t})}{(k-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{ke^t - e^{2t} + e^{2t}}{(k-e^t)^2}\right]_{t=0}$$

$$\mu_1' = \left[\frac{ke^t}{(k-e^t)^2}\right]_{t=0} \longrightarrow (1)$$

$$\mu_1' = \boxed{\text{Mean} = \frac{k}{(k-1)^2}}$$

(iii) Variance:

Put $r = 2$,

$$\mu_2' = \left[\frac{d^2}{dt^2}\left(\frac{e^t}{k-e^t}\right)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{e^t}{k-e^t}\right)\right)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{ke^t}{(k-e^t)^2}\right)\right]_{t=0} \quad [\text{by } (1)]$$

$$= \left[\frac{(k-e^t)^2(ke^t) - ke^t[2(k-e^t)(-e^t)]}{(k-e^t)^4}\right]_{t=0}$$

$$= \frac{(k-1)^2 k - k[2(k-1)(-1)]}{(k-1)^4}$$

$$= \frac{(k^2+1-2k)k + 2k(k-1)}{(k-1)^4}$$

$$= \frac{k^3+k-2k^2+2k^2-2k}{(k-1)^4}$$

$$\mu_2' = \frac{k^3+k-2k}{(k-1)^4}$$

$$\boxed{\mu_2' = \frac{k^3-k}{(k-1)^4}}$$

$$\text{variance} = \mu_2' - (\mu_1')^2$$

$$= \frac{k^3-k}{(k-1)^4} - \left(\frac{k}{(k-1)^2}\right)^2$$

$$= \frac{k^3-k}{(k-1)^4} - \frac{k^2}{(k-1)^4}$$

$$= \frac{k^3-k-k^2}{(k-1)^4}$$

$$\boxed{\text{variance} = \frac{k^3-k^2-k}{(k-1)^4}}$$

2) Find the MGF for the distribution

$$f(x) = \begin{cases} 2/3 & \text{at } x=1 \\ 1/3 & \text{at } x=2 \\ 0 & \text{otherwise} \end{cases}$$

(i) Mean (ii) variance

Soln:

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= e^{t} p(1) + e^{2t} p(2)$$

$$= e^{t} \cdot 2/3 + e^{2t}\, 1/3$$

$$= \frac{2}{3} e^{t} + \frac{1}{3} e^{2t}$$

$$\boxed{M_X(t) = \frac{2}{3} e^{t} + \frac{1}{3} e^{2t}}$$

(i) Mean:

$$\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{2}{3} e^{t} + \frac{1}{3} e^{2t}\right)\right]_{t=0}$$

$$= \left[\frac{2}{3} e^{t} + \frac{1}{3} e^{2t} \cdot 2\right]_{t=0}$$

$$= \frac{2}{3} + \frac{2}{3}$$

$$\boxed{\mu_1' = \frac{4}{3}} \quad \text{i.e.,} \quad \boxed{\text{Mean} = 4/3}$$

(ii) Variance:

$$\mu_2' = \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{2e^t}{3} + \frac{2e^{2t}}{3}\right)\right]_{t=0}$$

$$= \left[\frac{2e^t}{3} + \frac{4e^{2t}}{3}\right]_{t=0}$$

$$= \frac{2}{3} + \frac{4}{3}$$

$$= \frac{6}{3}$$

$$\boxed{\mu_2' = 2}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= 2 - \left(\frac{4}{3}\right)^2$$

$$= 2 - \frac{16}{9}$$

$$= \frac{18-16}{9}$$

$$\boxed{\text{Var} = \frac{2}{9}}$$

3) A R.V X has density function (3.35)

$$f(x) = \begin{cases} 2e^{-2x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

Obtained the (i) MGF (ii) four moments about the origin. (iii) Mean and variance.

Soln:

(i) M.G.F

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tX} f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tX}\, 2 \cdot e^{-2x}\, dx$$

$$= 2\int_{0}^{\infty} e^{tx} e^{-2x}\, dx$$

$$= 2\int_{0}^{\infty} e^{(t-2)x}\, dx$$

$$= 2\int_{0}^{\infty} e^{-(2-t)x}\, dx$$

$$= 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_{0}^{\infty}$$

$$= \frac{-2}{(2-t)}(0-1)$$

$$\boxed{M_X(t) = \frac{2}{2-t}}$$

(ii) $M_X(t) = \dfrac{2}{2-t}$

$$= \left(\frac{2-t}{2}\right)^{-1}$$

$$= \left(\frac{2}{2} - \frac{t}{2}\right)^{-1}$$

$$= \left(1 - \frac{t}{2}\right)^{-1}$$

$$= 1 + \frac{t}{2} + \left(\frac{t}{2}\right)^2 + \cdots + \left(\frac{t}{2}\right)^r + \cdots \longrightarrow (1)$$

$$= \frac{1}{0!} \times 0! + \frac{t}{2 \times 1!} \cdot 1! + \left(\frac{t}{2}\right)^2 \cdot \frac{2!}{2!} + \cdots + \left(\frac{t}{2}\right)^r \frac{r!}{r!} + \cdots$$

r^{th} moment about origin $\Big\}$ $= \mu_r' = r! \times \text{coeff. of } \dfrac{t^r}{r!}$

$$\boxed{\mu_r' = \frac{r!}{2^r}}$$

Put $r = 1, 2, 3, 4$ in $\dfrac{r!}{2^r}$

$\boxed{\mu_1' = \frac{1}{2}}$, $\mu_2' = \dfrac{2!}{4} = \dfrac{2 \times 1}{4} = \dfrac{1}{2} \Rightarrow \boxed{\mu_2' = 1/2}$

$\mu_3' = \dfrac{3!}{8} = \dfrac{3 \times 2 \times 1}{8} = \dfrac{3}{4} \Rightarrow \boxed{\mu_3' = \frac{3}{4}}$

$\mu_4' = \dfrac{4!}{16} = \dfrac{4 \times 3 \times 2 \times 1}{16} = \dfrac{3}{2} \Rightarrow \boxed{\mu_4' = \frac{3}{2}}$

(iii) Mean:

$$\mu_1' = 1/2$$

Variance:

$$\text{variance} = \mu_2' - (\mu_1')^2$$
$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$
$$= \frac{1}{2} - \frac{1}{4}$$
$$= \frac{4-2}{8} = \frac{2}{8} = \frac{1}{4}$$

$$\boxed{\text{var} = 1/4}$$

4) Find the m.g.f, mean and variance of the distribution whose p.m.f is

$$p(x) = \begin{cases} q^x p, & x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

given $p + q = 1$

Soln:

(i) M.G.F:

$$M_X(t) = E(e^{tX}) = \sum e^{tx} p(x)$$
$$= \sum_{x=0}^{\infty} e^{tx} q^x p$$
$$= e^0 q^0 p + e^t q^1 p + e^{2t} q^2 p + e^{3t} q^3 p + \dots$$

$$= p + e^t q p + e^{2t} q^2 p + e^{3t} q^3 p + \cdots$$

$$= p\left[1 + e^t q + (e^t q)^2 + (e^t q)^3 + \cdots\right]$$

$$= p(1 - qe^t)^{-1}$$

$$\boxed{M_X(t) = \frac{p}{1 - qe^t}}$$

(ii) <u>Mean</u>:

$$\mu_1' = \left\{\frac{d}{dt}\left[\frac{p}{1-qe^t}\right]\right\}_{t=0}$$

$$= \left[\frac{(1-qe^t)(0) - p(-qe^t)}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{pqe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$= \frac{pq}{(1-q)^2}$$

$$= \frac{pq}{p^2}$$

$$\boxed{\mu_1' = \frac{q}{p}}$$

(iii) <u>variance</u>:

$$\mu_2' = \left[\frac{d^2}{dt^2}\left(\frac{p}{1-qe^t}\right)\right]$$

$$= \left[\frac{d}{dt}\left(\frac{pqe^t}{(1-qe^t)^2}\right)\right]_{t=0}$$

$$= \left[\frac{(1-qe^t)^2\, pqe^t - \left[pqe^t \cdot 2(1-qe^t)\cdot(-qe^t)\right]}{(1-qe^t)^4}\right]_{t=0}$$

$$= \frac{(1-q)^2\, pq - \left[pq \cdot 2(1-q)(-q)\right]}{(1-q)^4}$$

$$= \frac{p^3q + 2p^2q^2}{p^4} = p^2q\left(\frac{p+2q}{p^4}\right)$$

$$\boxed{\mu_2' = \frac{q(p+2q)}{p^2}}$$

$$\text{variance} = \mu_2' - (\mu_1')^2$$

$$= \frac{q(p+2q)}{p^2} - \frac{q^2}{p^2}$$

$$= \frac{qp + 2q^2 - q^2}{p^2}$$

$$= \frac{qp+q^2}{p^2} = \frac{q(p+q)}{p^2}$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

//

Co-Variance :

Definition

If X and Y are two R.V, then the co-variance between them is $\boxed{Cov(X,Y) = E(XY) - E(X)E(Y)}$.

Results on covariance :

(i) $cov(aX, bY) = ab\, cov(X,Y)$

(ii) $cov(X+a, Y+b) = cov(X,Y)$

(iii) $cov(aX+b, cY+d) = ac\, cov(X,Y)$

(iv) $cov(X+Y, Z) = cov(X,Z) + cov(Y,Z)$

(v) $cov(aX+bY, cX+dY) = ac\, var(X) + bd\, var(Y) + (ad+bc)\, cov(X,Y)$

Note :

(i) If RV X & Y are independent, then

$$\boxed{E(XY) = E(X)\cdot E(Y)}$$

$$\Rightarrow \boxed{cov(X,Y) = 0}$$

But the converse is not true.

(ii) $\boxed{E(XY) = \sum_x \sum_y xy\, P(x,y)}$, where X & Y are not an independent variables

Variances :

If X and Y are R.V, then

$$cov(X,X) = Var(X) = \sigma_X^2 \text{ and}$$

$$cov(Y,Y) = Var(Y) = \sigma_Y^2$$

Results on Variance:

(i) $\text{var}(ax+by) = a^2\,\text{var}(x) + b^2\,\text{var}(Y) + 2ab\,\text{cov}(X,Y)$

Proof:

W.K.T, $V(X) = E(X^2) - [E(X)]^2$

$$V(ax+bY) = E[(ax+bY)^2] - [E(ax+bY)]^2 \longrightarrow (1)$$

Now, $E(ax+bY) = aE(X) + bE(Y)$

$$[E(ax+bY)]^2 = [aE(X) + bE(Y)]^2$$

$$[E(ax+bY)]^2 = a^2[E(X)]^2 + b^2[E(Y)]^2 + 2ab\,E(X).E(Y) \longrightarrow (2)$$

$$E[(ax+by)^2] = E[a^2x^2 + b^2y^2 + 2abxy]$$

$$E[(ax+by)^2] = a^2E(X^2) + b^2E(Y^2) + 2ab\,E(XY) \longrightarrow (3)$$

Substitute (3) & (2) in (1),

$$\text{var}(ax+bY) = a^2E(X^2) + b^2E(Y^2) + 2ab\,E(XY) - [a^2[E(X)]^2 + b^2[E(Y)]^2 + 2ab\,E(X)E(Y)]$$

$$= a^2[E(X^2) - [E(X)]^2] + b^2[E(Y^2) - [E(Y)]^2] + 2ab[E(XY) - E(X)E(Y)]$$

$$\boxed{\text{var}(ax+bY) = a^2\,\text{var}(X) + b^2\,\text{var}(Y) + 2ab\,\text{cov}(X,Y)}$$

Hence Proved.

(ii) If x and y are independent, then

$$Var(X \pm Y) = Var(X) + Var(Y)$$

Since $cov(X, Y) = 0$

(iii) $Var(aX + b) = a^2 \, Var(X)$

Derivation:

If X and Y are Independent r.v. Find covariance b/w $X+Y$ and $X-Y$.

Soln:

$$Cov(X+Y, X-Y) = E[(X+Y)(X-Y)] - E(X+Y)\,E(X-Y)$$

$$= E(X^2 - Y^2 + XY - XY) - [E(X) + E(Y)][E(X) - E(Y)]$$

$$= E(X^2 - Y^2) - \left[[E(X)]^2 - [E(Y)]^2\right]$$

$$= E(X^2) - E(Y^2) - [E(X)]^2 + [E(Y)]^2$$

$$= \left\{E(X^2) - [E(X)]^2\right\} - \left\{E(Y^2) - [E(Y)]^2\right\}$$

$$\boxed{Cov(X+Y, X-Y) = Var(X) - Var(Y)}$$

Problems:

1) If the independent R.V, X and Y have the variances 36 and 16 respectively; Find the $cov(X+Y, X-Y)$.

Soln: W.K.T,

$$cov(X+Y, X-Y) = var(X) - var(Y)$$
$$= 36 - 16$$
$$\boxed{cov(X+Y, X-Y) = 20}$$

2) If X has mean 4 and variance 9, while Y has mean -2 and variance 5 and the two are independent, find (a) $E(XY)$ (b) $E(XY^2)$

Soln:

Given, $E(X) = 4$, $E(Y) = -2$

$V(X) = 9$, $V(Y) = 5$

(a) Given: X & Y are independent.

$$E(XY) = E(X) \cdot E(Y)$$
$$= 4(-2)$$
$$\boxed{E(XY) = -8}$$

(b) $E(XY^2) = E(X)\, E(Y^2)$

Now, $var(Y) = E(Y^2) - [E(Y)]^2$

$$E(Y^2) = var(Y) + [E(Y)]^2$$
$$= 5 + [(-2)]^2$$
$$= 5+4$$
$$\boxed{E(Y^2) = 9}$$

Then, $E(XY^2) = E(X) \cdot E(Y^2)$
$$= 4(9)$$
$$\boxed{E(XY^2) = 36}$$

3) Let X_1 and X_2 have the joint pmf

$$P(x_1, x_2) = \frac{x_1 + 2x_2}{18}, \quad x_1 = 1, 2 \; ; \; x_2 = 1, 2 .$$

Find the $\text{cov}(X_1, X_2)$.

Soln: The joint pmf is

	$X_2 \backslash X_1$	1 ($P(X_1=1)$)	2 ($P(X_1=2)$)	$P(X_2)$
$P(X_2=1)$	1	3/18	4/18	7/18
$P(X_2=2)$	2	5/18	6/18	11/18
	$P(X_1)$	8/18	10/18	18/18

Marginal distribution function of X_1:

x_1	1	2
$P(X=x_1)$	8/18	10/18

Marginal distribution function of X_2

X_2	1	2
$P(X=X_2)$	$7/18$	$11/18$

$$E(X_1) = \Sigma X_1 P(X_1)$$

$$= 1 \cdot \left(\frac{8}{18}\right) + 2\left(\frac{10}{18}\right)$$

$$\boxed{E(X_1) = \frac{28}{18}}$$

$$E(X_2) = \Sigma X_2 P(X_2)$$

$$= 1\left(7/18\right) + 2\left(11/18\right)$$

$$\boxed{E(X_2) = \frac{29}{18}}$$

$$E(X_1 X_2) = \sum_{X_1}\sum_{X_2} X_1 X_2 P(X_1, X_2)$$

$$= \left(1 \cdot 1 \cdot 3/18\right) + \left(2 \cdot 1 \cdot 4/18\right) + \left(1 \cdot 2 \cdot 5/18\right) + \left(2 \cdot 2 \cdot 6/18\right)$$

$$= \frac{3}{18} + \frac{8}{18} + \frac{10}{18} + \frac{24}{18}$$

$$\boxed{E(X_1 . X_2) = \frac{45}{18}}$$

$$Cov(X_1, X_2) = E(X_1 X_2) - E(X_1) E(X_2)$$

$$= \frac{45}{18} - \frac{28}{18} \cdot \frac{29}{18}$$

$$= \frac{45}{18} - \frac{812}{18 \times 18}$$

$$= \frac{810-812}{18 \times 18} = \frac{-2}{324} = \frac{-1}{162}$$

$$\text{cov}(X_1, X_2) = -\frac{1}{162}$$

4) Let the Joint probability distribution of X & Y given by

Y \ X	-1	0	1	$P(Y=y)$
-1	1/6	1/3	1/6	4/6
0	0	0	0	0
1	1/6	0	1/6	2/6
$P(X=x)$	2/6	1/3	2/6	

(i) check whether X and Y are independent (or) not.

(ii) Find cov (X, Y)

Soln:

Marginal distribution of X is

X	-1	0	1
$P_X(x)$	2/6	1/6	2/6

Marginal distribution of Y is

Y	-1	0	1
$P_Y(y)$	4/6	0	2/6

Here, $P(-1,-1) = 1/6$

But $P_X(-1) = 2/6$

$P_Y(-1) = 4/6$

$$P(-1,-1) \neq P_X(-1)\, P_Y(-1)$$

$$\frac{1}{6} \neq \frac{8}{36}$$

$\therefore$ X & Y are not independent.

$$E(X) = \Sigma\, X\, P(X)$$

$$= -1(2/6) + 0(1/3) + 1(2/6)$$

$$\boxed{E(X) = 0}$$

$$E(Y) = \Sigma\, Y\, P(Y)$$

$$= -1(4/6) + 0(0) + 1(2/6)$$

$$\boxed{E(Y) = -2/6}$$

$$E(XY) = \sum_X \sum_Y XY\, P(X,Y)$$

$$= (-1)(-1)(1/6) + (0)(1)(1/3) + (1)(-1)\cdot(1/6)$$

$$(-1)(1)(1/6) + (0)(1)(0) + (1)(1)(1/6)$$

$$= \frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{6} = 0$$

$$\boxed{E(XY) = 0}$$

$$\text{cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$$
$$= 0 - 0(-2/6)$$
$$= 0$$

$$\boxed{\text{cov}(X,Y) = 0}$$

5) X and Y are discrete R.Vs. If $\text{Var}(X) = \text{Var}(Y) = \sigma^2$, $\text{Cov}(X,Y) = \frac{\sigma^2}{2}$, Find $\text{var}(2X-3Y)$

Soln:

W.K.T, $\text{Var}(aX+bY) = a^2\text{var}(X) + b^2\text{var}(Y) + 2ab\,\text{cov}(X,Y)$

$$\text{var}(2X-3Y) = 4\,\text{var}(X) + 9\,\text{var}(Y) + \cancel{2}(2)(-3)\cdot\frac{\sigma^2}{\cancel{2}}$$
$$= 4\sigma^2 + 9\sigma^2 - 6\sigma^2$$

$$\boxed{\text{var}(2X-3Y) = 7\sigma^2}$$

6) $f(x,y) = \begin{cases} 8xy, & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$

Find the $\text{cov}(X,Y)$

Soln:

(i) Marginal distribution function of X.

N.K.T, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_0^1 8xy\,dy$$

$$= 8x \int_0^1 y\,dy$$

$$= 8x \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{8x}{2}[1-0]$$

$$\boxed{f_X(x) = 4x, \quad 0 \le x \le 1}$$

(ii) Marginal distribution function of Y:

W.K.T, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_0^1 8xy\,dx$$

$$= 8y \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{8y}{2}[1-0]$$

$$\boxed{f_Y(y) = 4y, \quad 0 \le y \le 1}$$

(iii) W.K.T, $E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$

$$= \int_0^1 x(4x)\,dx$$

$$= \left[\frac{4x^3}{3}\right]_0^1$$

$$= \left[\frac{4}{3} - 0\right] \Rightarrow \boxed{E(X) = 4/3}$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

$$= \int_{0}^{1} y\,(4y)\, dy$$

$$= \left[\frac{4y^3}{3}\right]_0^1$$

$$= \left[\frac{4}{3} - 0\right] = 4/3$$

$$\boxed{E(Y) = 4/3}$$

(iv) $$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_{0}^{1}\int_{0}^{1} xy\,(8xy)\, dx\, dy$$

$$= 8\int_{0}^{1}\int_{0}^{1} x^2 y^2\, dx\, dy$$

$$= 8\int_{0}^{1} y^2 \left(\frac{x^3}{3}\right)_0^1 dy$$

$$= \frac{8}{3}\int_{0}^{1} y^2 (1-0)\, dy$$

$$= \frac{8}{3}\int_{0}^{1} y^2\, dy$$

$$= \frac{8}{3}\left(\frac{y^3}{3}\right)_0^1 = \frac{8}{9}(1-0) = 8/9$$

$$\boxed{E(XY) = 8/9}$$

(v) $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$$= \frac{8}{9} - \frac{4}{3}\left(\frac{4}{3}\right)$$

$$= \frac{8}{9} - \frac{16}{9}$$

$$\boxed{\text{cov}(x,y) = -\frac{8}{9}}$$

7) The Joint p.d.f of the two dimensional R.V's (x,y) is
$$f(x,y) = \begin{cases} 2-x-y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the cov (x,y).

Soln:

(i) Marginal distribution function of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 (2-x-y)\,dy$$

$$= \left(2y - xy - \frac{y^2}{2}\right)_0^1$$

$$= 2 - x - 1/2$$

$$\boxed{f_X(x) = \frac{3}{2} - x, \quad 0 \le x \le 1}$$

(ii) Marginal distribution function of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^1 (2-x-y)\,dx$$

$$= \left(2x - \frac{x^2}{2} - xy\right)_0^1$$

$$= 2 - \frac{1}{2} - y$$

$$\boxed{f_Y(y) = \frac{3}{2} - y, \quad 0 \leq y \leq 1}$$

(iii) $$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= \int_0^1 x\left(3/2 - x\right)dx$$

$$= \int_0^1 \left(\frac{3x}{2} - x^2\right)dx$$

$$= \left(\frac{3x^2}{4} - \frac{x^3}{3}\right)_0^1$$

$$= \frac{3}{4} - \frac{1}{3}$$

$$\boxed{E(X) = \frac{5}{12}}$$

(iv) $E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$

$= \int_0^1 y\left(\frac{3}{2} - y\right) dy$

$= \int_0^1 \left(\frac{3y}{2} - y^2\right) dy$

$= \left(\frac{3y^2}{4} - \frac{y^3}{3}\right)_0^1$

$= \frac{3}{4} - \frac{1}{3}$

$\boxed{E(Y) = \frac{5}{12}}$

(v) $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$= \int_0^1 \int_0^1 xy(2-x-y)\, dx\, dy$

$= \int_0^1 \int_0^1 (2xy - x^2y - xy^2)\, dx\, dy$

$= \int_0^1 \left(\frac{2yx^2}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right)_0^1 dy$

$= \int_0^1 \left(y - \frac{y}{3} - \frac{y^2}{2}\right) dy$

$= \left(\frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6}\right)_0^1$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{6}$$

$$= \frac{3-1-1}{6}$$

$$\boxed{E(XY) = \frac{1}{6}}$$

(vi) $\text{cov}(X,Y) = E(XY) - E(X)\,E(Y)$

$$= \frac{1}{6} - \left(\frac{5}{12}\right)\left(\frac{5}{12}\right)$$

$$= \frac{1}{6} - \frac{25}{144}$$

$$= \frac{24-25}{144}$$

$$\boxed{\text{cov}(X,Y) = -\frac{1}{144}}$$

H.W.

8) The joint p.d.f of a two dimensional r.v is given by $f(x,y) = \frac{1}{3}(x+y)$, $0 \le x \le 1$, $0 \le y \le 2$. Find the $\text{cov}(x,y)$

Ans:

$$f_X(x) = \frac{2}{3}(x+1), \quad 0 \le x \le 1$$

$$f_Y(y) = \frac{1}{6}[1+2y], \quad 0 \le y \le 2$$

$$E(X) = 5/9, \quad E(Y) = 11/9$$

$$E(XY) = 2/3, \quad \text{cov}(X,Y) = -\frac{1}{81}$$

Unit - IV

Theoretical Discrete Distributions :

Binomial distribution :

Consider an experiment with the following Properties :

(i) The experiment consists of a fixed number 'n' of Bernoulli's trials, i.e., trials such that the result is either success (s) or failure (f).

(ii) The trials are identical and independent and therefore, the probability of success 'p' remains the same from trial to trial.

(iii) The random variable X denotes the number of successes obtained in n trials.

Definition :

A random variable X is said to follow a binomial distribution if it assumes only non-negative values with probability mass function.

$$\boxed{P(X=x) = nC_x\, p^x q^{n-x}}$$ where $x = 0, 1, 2, \ldots, n$ & $q = 1-p$.

Here the two independent constants n and p are the parameters of binomial distribution.

Binomial theorem:

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^{n-x} b^x$$

$$\boxed{(a+b)^n = \sum_{x=0}^{n} nC_x \, a^{n-x} b^x} \longrightarrow (I)$$

where, $nC_x = \dfrac{n!}{(n-x)!\,x!}$

Moment Generating function: (MGF)

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (e^t p)^x q^{n-x}$$

$$= (q+pe^t)^n \qquad [\text{by } (I)]$$

$$\therefore \boxed{M_X(t) = (q+pe^t)^n} \qquad (\because a = q,\ b = pe^t)$$

Mean and variance using MGF:

Mean: $\left[M_X'(t)\right]_{t=0} = \left[\dfrac{d}{dt} M_X(t)\right]_{t=0}$

$$= \left\{\frac{d}{dt}\left[q+pe^t\right]^n\right\}_{t=0}$$

$$= \left[n(q+pe^t)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= \left[npe^{t}(q+pe^{t})^{n-1}\right]_{t=0}$$

$$= np(q+p)^{n-1}$$

$$= np(1)^{n-1}$$

$\boxed{\text{Mean} = np}$ (or) $\boxed{\mu_1' = np}$

$$\mu_2' = \left[M_x''(t)\right]_{t=0} = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[npe^{t}(q+pe^{t})^{n-1}\right]\right\}_{t=0}$$

$$= np\left[\frac{d}{dt}\left[e^{t}(q+pe^{t})^{n-1}\right]\right]_{t=0}$$

$$= np\left[e^{t}\left\{(n-1)(q+pe^{t})^{n-2}pe^{t}\right\} + (q+pe^{t})^{n-1}e^{t}\right]_{t=0}$$

$$= np\left[(n-1)(q+p)^{n-2}\cdot p + (q+p)^{n-1}\right]$$

$$= np\left[(n-1)p+1\right]$$

$$= np\left[np-p+1\right]$$

$$= np\left[np+q\right] \quad (\because 1-p=q)$$

$$\boxed{\mu_2' = n^2p^2 + npq}$$

Variance:

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= n^2p^2 + npq - (np)^2 = n^2p^2 + npq - n^2p^2$$

$$\mu_2 = npq$$

$\boxed{\text{Variance} = npq}$

Additive property of Binomial Distribution: (4.4)

If $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$ are independent random variables, then

$X+Y \sim B(n_1+n_2, p)$

Proof:

$$M_X(t) = (q+pe^t)^{n_1}$$
$$M_Y(t) = (q+pe^t)^{n_2}$$

X and Y are independent.

$$\therefore \; M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$
$$= (q+pe^t)^{n_1} \cdot (q+pe^t)^{n_2}$$
$$= (q+pe^t)^{n_1+n_2}$$

which is the m.g.f of a binomial variate with parameters (n_1+n_2, p)

Problems:

1) Comment on the following: X follows a Binomial distribution with mean 3 and Var 4.

Soln:

$$\text{Mean} = np = 3$$

$$\text{Var} = npq = 4$$

$$\Rightarrow \frac{npq}{np} = \frac{4}{3}$$

$$\Rightarrow q = \frac{4}{3} > 1 \quad (\text{Not Possible}) \; (\because p+q=1)$$

Since probability does not exceed unity

4.5

2) For a B.D, mean = 6 and S.D = $\sqrt{2}$.

Find the first two terms of the distribution.

Soln:

Mean = $np = 6$, Var = $npq = 2$

$$\Rightarrow \frac{npq}{np} = \frac{2}{6} \Rightarrow \boxed{q = \frac{1}{3}}$$

$$p = 1 - q = 1 - 1/3 = 2/3$$

$$\boxed{p = 2/3}$$

Now, $np = 6$

$$n(2/3) = 6$$

$$n = 6 \cdot (3/2)$$

$$\boxed{n = 9}$$

Now, the pmf is $P(X=x) = 9C_x \, (2/3)^x \, (1/3)^{9-x} \longrightarrow (1)$

First term: put $x = 0$, in (1)

$$P(X=0) = 9C_0 \, (2/3)^0 \, (1/3)^{9-0}$$

$$= 1 (1) (1/3)^9$$

$$= (1/3)^9$$

Second term: Put $x = 1$, in (1)

$$P(X=1) = 9C_1 \, (2/3)^1 \, (1/3)^{9-1}$$

$$= 9 (2/3) (1/3)^8$$

$$= 6 (1/3)^8$$

3) The mean of a B.D is 20 and S.D = 4. Find the Parameters of the distribution.

Soln:

$$\text{Mean} = np = 20, \quad \text{Var} = npq = 16$$

$$\Rightarrow \frac{npq}{np} = \frac{16}{20} = \frac{4}{5}$$

$$\boxed{q = \frac{4}{5}}$$

$$p = 1 - q$$

$$= 1 - \frac{4}{5} = \frac{5-4}{5}$$

$$\boxed{p = \frac{1}{5}}$$

$$np = 20$$

$$n(1/5) = 20$$

$$n = 20(5) = 100$$

$$\boxed{n = 100}$$

The parameters $\boxed{n = 100 \ \& \ p = 1/5}$ respectively

4) Determine the B.D whose mean is 9, whose S.D is 3/2.

Soln:

$$\text{Mean} = np = \boxed{9}$$

$$\text{Var} = (S.D)^2 = (3/2)^2 = \boxed{9/4}$$

$$\Rightarrow \frac{npq}{np} = \frac{9}{4}/9 = \frac{9}{4} \cdot \frac{1}{9} = \boxed{1/4} \Rightarrow \boxed{q = 1/4}$$

$p = 1 - q$

$= 1 - 1/4$

$\boxed{p = 3/4}$

$np = 9$

$n(3/4) = 9$

$n = 9 \times \frac{4}{3} = 3(4) = 12$

$\boxed{n = 12}$

$\boxed{P(X = x) = 12C_x \left(3/4\right)^x \left(1/4\right)^{12-x}}$ where $x = 0, 1, 2, \ldots, 12$

5) For a R.V X, $M_X(t) = \frac{1}{81}(e^t + 2)^4$, find $P(X \leq 2)$.

Soln: Given, $M_X(t) = \frac{1}{81}(e^t + 2)^4$

$= \frac{1}{3^4}(e^t + 2)^4 \quad (\because 81 = 3^4)$

$= \left(\frac{e^t}{3} + \frac{2}{3}\right)^4$

$M_X(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^4 \longrightarrow (1)$

W.K.T, $M_X(t) = (q + pe^t)^n \longrightarrow (2)$

Comparing (1) & (2), we get

$\boxed{q = 2/3 \; ; \; p = \frac{1}{3} \; ; \; n = 4}$

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= 4C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{4-0} + 4C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{4-1} +$

$4C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{4-2}$

$= 1(1)(2/3)^4 + 4(1/3)(2/3)^3 + 6(1/3)^2(2/3)^2$

$= \frac{16}{81} + \frac{4}{3}\left(\frac{8}{27}\right) + 6\left(\frac{1}{9}\right)\left(\frac{4}{9}\right)$

$= \frac{16+32+24}{81} = \frac{72}{81}$

$\boxed{P(X \le 2) = 0.8389}$

6) The mean and variance of a Binomial variate is 4 and $\frac{4}{5}$ respectively. Find $P(X \ge 1)$

Soln:

Mean $= np = 4$

var $= npq = \frac{4}{5}$

$\frac{npq}{np} = \frac{4/5}{4} = \boxed{\frac{1}{5}} \Rightarrow \boxed{q = 1/5}$

$p = 1 - q$

$= 1 - 1/5$

$\boxed{p = 4/5}$

$np = 4$

$n(4/5) = 4$

$n = 4(5/4) = 5 \Rightarrow \boxed{n = 5}$

$P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - 5C_0 \left(4/5\right)^0 \left(1/5\right)^{5-0}$

$= 1 - \frac{1}{3125}$

$= \frac{3125 - 1}{3125}$

$= \frac{3124}{3125}$

$\boxed{P(X \geq 1) = 0.9997}$

7) For a B.D, mean is 4 & var is 2. Find the first probability of getting

(i) atleast 2 successes

(ii) atmost 2 successes

(iii) find $P(5 \leq X \leq 7)$

Soln:

Mean $= np = 4$

Var $= npq = 2$

$\frac{npq}{np} = \frac{2}{4}$

$\Rightarrow \boxed{q = \frac{1}{2}}$

$p = 1 - 1/2 = 1/2$

$\boxed{p = 1/2}$

$np = 4$ 4.10

$n(1/2) = 4$

$\boxed{n = 8}$

(i) atleast 2 successes : $p(X \geq 2)$.

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - [P(X=0) + P(X=1)]$$
$$= 1 - \left[8C_0 (1/2)^0 (1/2)^{8-0} + 8C_1 (1/2)^1 (1/2)^{8-1}\right]$$
$$= 1 - [1 \cdot 1 (1/2)^8 + 8(1/2)(1/2)^7]$$
$$= 1 - \left[\frac{1}{256} + \frac{8}{256}\right]$$
$$= 1 - 9/256$$
$$= \frac{256-9}{256} = \frac{247}{256}$$

$$\boxed{P(X \geq 2) = 0.9648}$$

(ii) Atmost 2 successes : $P(X \leq 2)$.

$$\therefore P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$
$$= \frac{9}{256} + 8C_2 (1/2)^2 (1/2)^6$$
$$= \frac{9}{256} + 28 (1/4)(1/64)$$

$$= \frac{9+28}{256}$$

$$= \frac{37}{256}$$

$$\boxed{P(X \le 2) = 0.1445}$$

(iii) Find $P(5 \le X \le 7) = P(X=5) + P(X=6) + P(X=7)$

$$= 8C_5 (1/2)^5 (1/2)^3 + 8C_6 (1/2)^6 (1/2)^2 + 8C_7 (1/2)^7 (1/2)^1$$

$$= 56(1/32)(1/8) + 28(1/64)(1/4) + 8(1/128)(1/2)$$

$$= \frac{28+14+4}{128}$$

$$= \frac{46}{128}$$

$$\boxed{P(5 \le X \le 7) = 0.3594}$$

8) A binomial variable X satisfies the relation $9P(X=4) = P(X=2)$ when $n=6$. Find the parameter p of the B.D.

Soln: W.K.T, $P(X=x) = nC_x \, p^x q^{n-x}$

$$P(X=4) = 6C_4 \, p^4 q^{6-4}$$

$$= \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} p^4 q^{6-4}$$

$$\boxed{P(X=4) = 15 p^4 q^2}$$

$$P(X=2) = 6C_2 \, p^2 q^{6-2}$$

$$P(X=2) = \frac{6\times5}{1\times2}\, p^2 q^4$$

$$\boxed{P(X=2) = 15\, p^2 q^4}$$

Given: $9P(X=4) = P(X=2)$

$$9\left(15 p^4 q^2\right) = 15 p^2 q^4$$

$$9p^2 = q^2$$

$$9p^2 = (1-p)^2$$

$$9p^2 = 1+p^2-2p$$

$$8p^2+2p-1=0$$

-8

$4 \quad -2 \quad 2$

$$8p^2+4p-2p-1=0$$

$$4p(2p+1)-1(2p+1)=0$$

$$(4p-1)(2p+1)=0$$

$4p-1=0 \qquad 2p+1=0$

$4p=1 \qquad 2p=-1$

$p=1/4 \qquad p=-1/2$

$\therefore \boxed{p=1/4}$ // $\quad$ ($\because p=-1/2$ is impossible)

9) A discrete R.V X has m.g.f $M_X(t) = \left(\frac{1}{4}+\frac{3}{4}e^t\right)^5$. Find $E(X)$, $Var(X)$ & $P(X=2)$.

Soln:

W.K.T, $\quad M_X(t) = (q+pe^t)^n \longrightarrow (1)$

Given, $\quad M_X(t) = \left(\frac{1}{4}+\frac{3}{4}e^t\right)^5 \longrightarrow (2)$

Comparing (1) & (2),

$q = \frac{1}{4}$, $p = \frac{3}{4}$, $n = 5$

Mean $= E(X) = np = 5(3/4) = \boxed{15/4}$

var(X) $= npq = 5 \cdot 3/4 \cdot 1/4 = 15/20 = \boxed{5/6}$

$P(X=2) = 5C_2 (3/4)^2 (1/4)^{5-2}$

$= \frac{5 \times 4}{1 \times 2} \left(\frac{9}{16}\right)\left(\frac{1}{4}\right)^3$

$P(X=2) = 10\left(\frac{9}{16}\right)\left(\frac{1}{64}\right) = \frac{45}{512} = \boxed{0.087789}$

10) In a B.D consisting of 5 independent trials, probabilities of 1 and 2 success are 0.4096 and 0.2048 respectively. Find $P(X=3)$.

Soln:

Given: Here $n = 5$

$P(X=1) = 0.4096$

$P(X=2) = 0.2048$

To find: $P(X=3)$

W.K.T, $P(X=x) = nC_x p^x q^{n-x}$

$P(X=1) = 5C_1 p^1 q^{5-1} = 0.4096$

$\Rightarrow \boxed{5pq^4 = 0.4096} \rightarrow (1)$

$$P(X=2) = 5C_2\, p^2 q^{5-2} = 0.2048$$

$$\Rightarrow \boxed{10p^2 q^3 = 0.2048} \rightarrow (2)$$

$$\frac{(1)}{(2)} = \frac{5pq^4}{10p^2q^3} = \frac{0.4096}{0.2048}$$

$$\frac{q}{2p} = 2$$

$$q = 4p$$

$$q = 4(1-q)$$

$$q = 4 - 4q$$

$$5q = 4$$

$$\boxed{q = 4/5}$$

$$p = 1-q$$

$$= 1 - 4/5$$

$$\boxed{p = 1/5}$$

$$P(X=3) = 5C_3 \left(1/5\right)^3 \left(4/5\right)^{5-3}$$

$$= \frac{5\times4\times3}{1\times2\times3}\left(\frac{1}{125}\right)\left(\frac{4}{5}\right)^2$$

$$= 10 \times \frac{1}{125} \times \frac{16}{25}$$

$$= \frac{32}{625}$$

$$\boxed{P(X=3) = 0.0512}$$

11) If, on an average, 9 ships out of 10 arrive safely to a port, obtain the mean and S.D of the number of ships returning safely out of 150 ships. 4.15

Soln:

If p is the probability of safe arrival

then $\boxed{p = \frac{9}{10}} \Rightarrow q = 1 - p$

$= 1 - 9/10$

$\boxed{q = 1/10}$

and $\boxed{n = 150}$

Mean $= np$

$= 150\,(9/10)$

$\boxed{\text{Mean} = 135}$

Variance $= npq$

$= 135 \times 1/10$

Var $= \boxed{13.5}$

$S.D = \sqrt{13.5}$

$\boxed{S.D = 3.674}$

12) A gun is fixed at a target from a certain distance. The probability of hitting the target is 0.2. Totally two bombs are enough to destroy the target. If six shells are aimed at the target,

find the probability of the target being destroyed. (4.16)

Soln:

Let p be the probability of shell hitting the target.

Here, $p = 0.2$

$q = 1 - p$

$= 1 - 0.2$

$\boxed{q = 0.8}$ and $\boxed{n = 6}$

Let X be the number of shells required to destroy the target.

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - [6C_0 (0.2)^0 (0.8)^6 + 6C_1 (0.2)^1 (0.8)^5]$$

$$= 1 - [1(1)(0.8)^6 + 6(0.2)(0.8)^5]$$

$$= 1 - [0.2621 + 0.3932]$$

$$= 1 - 0.6553$$

$$\boxed{P(X \geq 2) = 0.3447}$$

13) The overall percentage of failure in a certain examination is 40. What is the probability that out of a group of 6 candidates atleast 4 passes the examination.

Soln: Let p be the probability of passing the examination, then $q = 1-p$ is the probability of failures. (4.17)

$$\text{Here } q = \frac{40}{100} = \boxed{0.4}$$

$$p = 1 - 0.4 = \boxed{0.6}$$

$$n = 6$$

P(atleast four persons passes the examination)

i.e., $P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$

$$= 6C_4 (0.6)^4 (0.4)^{6-4} + 6C_5 (0.6)^5 (0.4)^{6-5} + 6C_6 (0.6)^6 (0.4)^{6-6}$$

$$= \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} (0.6)^4 (0.4)^2 + \frac{6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5} (0.6)^5 (0.4)^1 + (0.6)^6 (1)$$

$$= 15 (0.6)^4 (0.4)^2 + 6 (0.6)^5 (0.4) + (0.6)^6$$

$$= 0.31104 + 0.186624 + 0.046656$$

$$P(X \geq 4) = 0.54432$$

14) A company produces screws. It is known that 0.01 of total production is defective. The company sells the screws in packages of 10 and the company announces that money will

be given back if atleast 2 of the 10 Screws is defective. What proportion of the packages Sold must be replaced? Use Binomial distribution.

Soln:

If X be the no. of defective Screws in the package.

The probability that a Screw is defective $(p) = 0.01$

$\boxed{p = 0.01}$ & $q = 1 - p$

$\boxed{n = 10}$ $= 1 - 0.01$

$\boxed{q = 0.99}$

The probability that a package will have to be replaced.

P(atleast 2 out of 10 Screws is defective)

$$= P(X \geq 2)$$

$$= 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[10C_0 (0.01)^0 (0.99)^{10-0} + 10C_1 (0.01)^1 (0.99)^{10-1}\right]$$

$$= 1 - [1(1)(0.99)^{10} + 10(0.01)(0.99)^9]$$

$$= 1 - [0.9044 + 0.09135]$$

$$= 1 - [0.9957]$$

$$\boxed{P(X \geq 2) = 0.0043}$$

15) Six dice are thrown 729 times. How many times do you expect atleast 3 dice to show a five or a six?

Soln:

Success = getting 5 or 6 with 1 die

p = probability of getting 5 or 6 with one die.

$p = \frac{2}{6} = \frac{1}{3}$

$q = 1 - p$

$= 1 - \frac{1}{3}$

$\boxed{q = \frac{2}{3}}$

Rough

Event $A = 5$

Event $B = 6$

$P(A \cup B) = P(A) + P(B)$

$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

Let X be a r.v denoting the number of success when 6 dice are thrown once.

Given, $n = 6$.

Then $P(X = x) = 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$

P(atleast 3 dice show five or six) $= P(X \geq 3)$

$$= 1 - P(X < 3)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{6-0} + 6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{6-1} + 6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{6-2}\right]$$

$$= 1 - \left[1(1)\left(\frac{2}{3}\right)^6 + 6\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^5 + 15\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4\right]$$

$$= 1 - \left[\frac{64}{729} + \frac{192}{729} + \frac{240}{729}\right]$$

$$= 1 - \frac{496}{729}$$

$$= \frac{729 - 496}{729}$$

$$P(X \geq 3) = \frac{233}{729}$$

6 dice are thrown 729 times. Hence, the expected number of times atleast three dice show five or six.

$$= \frac{233}{729} \times 729$$

$$= 233 \text{ times}$$

16) The probability of a man hitting a target is $1/3$. How many times must he fire so that the probability of hitting the target atleast once is more than 90%.

Soln: Let X be the no. of times he hits the target.

$p = 1/3$ then $q = 1 - p$

$$= 1 - 1/3$$

$$\boxed{q = 2/3}$$

Then X follows a B.D, with $P(X=x) = nC_x\, p^x q^{n-x}$

(4.21)

Given: $P(X \geq 1) > 90\%$.

$$1 - P(X<1) > 0.9$$

$$-P(X<1) > 0.9 - 1$$

$$-P(X<1) > -0.1$$

$$P(X<1) < 0.1$$

$$P(X=0) < 0.1$$

$$nC_0 \left(\tfrac{1}{3}\right)^0 \left(\tfrac{2}{3}\right)^{n-0} < 0.1$$

$$1 \cdot 1 \cdot \left(\tfrac{2}{3}\right)^n < 0.1$$

$$\left(\tfrac{2}{3}\right)^n < 0.1$$

Taking 'log' on both sides

$$n \log 2/3 < \log 0.1$$

$$n(-0.1761) < -1$$

$$-0.1761\, n < -1$$

$$0.1761\, n > 1$$

$$n > \frac{1}{0.1761}$$

$$n > 5.6786$$

$$n \geq 6$$

$\therefore$ He must fire at least 6 times.

Poisson Distribution:

Definition:

If X is a discrete R.V that assumes the values $0, 1, 2, \ldots$ such that the probability mass function is given by

$$P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & ; x = 0, 1, 2, \ldots \\ & \quad \lambda > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

then X is said to follow a poisson distribution with parameter λ.

Poisson distribution is the limiting case of B.D:

Suppose in a B.D,

1) The no. of trials is indefinitely large.
 ie., $n \to \infty$.

2) p, the probability of success in each trial is very small. ie., $p \to 0$

3) $np\,(=\lambda)$ is finite and $p = \lambda/n$.
 $q = 1 - p = 1 - \lambda/n$

Now for binomial distribution,

$$P(X=x) = nC_x\, p^x q^{n-x}$$

$$= \frac{n!}{(n-x)!\,x!}\,p^{x} q^{n-x}$$

$$= \frac{\underbrace{1\cdot 2\cdot 3\cdot \ldots (n-x)}\,(n-x+1)\ldots (n-2)(n-1)\cdot n}{(n-x)!\;x!}\cdot p^{x} q^{n-x}$$

$$= \frac{\cancel{(n-x)!}\left\{[n-(x-1)]\ldots (n-2)(n-1)\cdot n\right\}}{\cancel{(n-x)!}\;x!}\,p^{x} q^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots [n-(x-1)]}{x!}\left(\frac{\lambda}{n}\right)^{x}\left(1-\lambda/n\right)^{n-x}$$

$$= \frac{\lambda^{x}}{x!}\, n^{x}\left[1\cdot(1-1/n)(1-2/n)\ldots\left(1-\left(\frac{x-1}{n}\right)\right)\right](1/n^{x})(1-\lambda/n)^{n-x}$$

$$P(X=x) = \frac{\lambda^{x}}{x!}\left[(1-1/n)(1-2/n)\ldots\left[1-\left(\frac{x-1}{n}\right)\right]\right](1-\lambda/n)^{n}(1-\lambda/n)^{-x}$$

Taking limits on both sides

$$\lim_{n\to\infty} P(X=x) = \lim_{n\to\infty}\left[\frac{\lambda^{x}}{x!}\left\{(1-1/n)(1-2/n)\ldots\left[1-\left(\frac{x-1}{n}\right)\right]\right\}\cdot (1-\lambda/n)^{n}(1-\lambda/n)^{-x}\right]$$

$$P(X=x) = \frac{\lambda^{x}}{x!}(1)\lim_{n\to\infty}(1-\lambda/n)^{n}$$

$$= \frac{\lambda^{x}}{x!}\lim_{n\to\infty}(1-\lambda/n)^{n}$$

$$= \frac{\lambda^{x}}{x!}e^{-\lambda} \qquad \left[\because \lim_{n\to\infty}(1-\lambda/n)^{n} = e^{-\lambda}\right]$$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$
$$\lambda > 0$$

which is the p.m.f of poisson distribution.

Moments about origin of poisson distribution:

(i) $\mu_1' = \lambda$

(ii) $\mu_2' = \lambda^2 + \lambda$

(iii) $\mu_3' = \lambda^3 + 3\lambda^2 + \lambda$

(iv) $\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$

Moment Generating Function (M.G.F), Mean & Variance:

W.K.T, $M_X(t) = E(e^{tX})$

$$= \Sigma e^{tx} P(X=x)$$
$$= \Sigma e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx}\lambda^{x}}{x!}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$
$$= e^{-\lambda}\left[\frac{(\lambda e^{t})^{0}}{0!} + \frac{(\lambda e^{t})^{1}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \ldots\right]$$

4.25

$$= e^{-\lambda}\left[1 + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right]$$

$$= e^{-\lambda} e^{\lambda e^t}$$ $$\left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$= e^{\lambda e^t - \lambda}$$ $$\left(\text{Here } x = \lambda e^t\right)$$

$$\boxed{M_X(t) = e^{\lambda(e^t - 1)}}$$

Mean:

$$M_X'(t) = \left[\frac{d}{dt}\left(M_X(t)\right)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{e^{\lambda(e^t-1)}\right\}\right]_{t=0}$$

$$= \left[e^{\lambda(e^t-1)} \lambda e^t\right]_{t=0}$$

$$= e^{\lambda(e^0-1)} \lambda e^0$$

$$= e^0 \lambda e^0$$

$$\boxed{\mu_1' = \text{Mean} = \lambda}$$

$$M_X''(t) = \left[\frac{d^2}{dt^2}\left(M_X(t)\right)\right]_{t=0}$$

$$= \frac{d}{dt}\left[e^{\lambda(e^t-1)} \cdot \lambda e^t\right]_{t=0}$$

$$= \lambda\left[e^t \cdot e^{\lambda(e^t-1)} \lambda e^t + e^{\lambda(e^t-1)} e^t\right]_{t=0}$$

$$= \lambda\left[e^{0}.e^{0}\lambda e^{0} + e^{0}e^{0}\right]$$

$$= \lambda(\lambda+1)$$

$$\boxed{\mu_2' = \lambda^2 + \lambda}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$\boxed{\text{Variance} = \lambda}$$

Recurrence Relation for the moments of the P.D:

$$\mu_{r+1} = \lambda\left[\frac{d}{d\lambda}\mu_r + r\,\mu_{r-1}\right]$$

Additive Property of Poisson Distribution:

If X and Y are independent poisson variates with parameters λ_1 and λ_2 respectively, then $X+Y$ is also a poisson variate with Parameter $\lambda_1 + \lambda_2$.

Proof:

X is a poisson variate with parameter λ_1

$$\therefore M_X(t) = e^{\lambda_1(e^t - 1)}$$

Y is a poisson variate with Parameter λ_2

$$\therefore M_Y(t) = e^{\lambda_2(e^t - 1)}$$

Now, $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

($\because X$ & Y are independent)

$$M_{X+Y}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$$

$$M_{X+Y}(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}$$

$\therefore X+Y$ is also a Poisson variate with parameter $\lambda_1+\lambda_2$.

So, it satisfy the additive property of P.D

Problems:

1) If X_1 and X_2 are independent Poisson variates, S.T $X_1 - X_2$ is not a poisson variate.

Soln:

Let X_1 and X_2 be independent poisson variates having parameters λ_1 & λ_2 respectively.

Now,

$$M_{X_1-X_2}(t) = M_{X_1}(t) \cdot M_{(-X_2)}(t)$$

$$= e^{\lambda(e^t-1)} \cdot e^{-\lambda(e^t-1)}$$

$$= M_{X_1}(t) \cdot M_{X_2}(-t)$$

[$\because M_{cX}(t) = M_X(ct)$] By the property of M.G.F

$$M_{X_1-X_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^{-t}-1)}$$

The above cannot be expressed in the form $e^{\lambda(e^t-1)}$.

$\therefore$ $X_1 - X_2$ is not a poisson variate.

2) If X is a Poisson R.V, such that $P(X=1) = \frac{3}{10}$, $P(X=2) = \frac{1}{5}$, find $P(X=0)$ & $P(X=3)$.

Soln:

Given: $P(X=1) = \frac{3}{10}$, $P(X=2) = \frac{1}{5}$

W.K.T, $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots,$
$\lambda > 0$

$$P(X=1) = \frac{e^{-\lambda}\lambda^1}{1!} = e^{-\lambda}.\lambda$$

$$\Rightarrow e^{-\lambda}.\lambda = \frac{3}{10} \rightarrow (1)$$

$$P(X=2) = \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-\lambda}\lambda^2}{2} = \frac{1}{5}$$

$$\Rightarrow e^{-\lambda}\lambda^2 = \frac{2}{5} \rightarrow (2)$$

$$\frac{(2)}{(1)} = \frac{e^{-\lambda}\lambda^2}{e^{-\lambda}\lambda} = \frac{2}{5} \times \frac{10}{3}$$

$$\boxed{\lambda = \frac{4}{3} = 1.3333}$$

To find:

(i) $$P(X=0) = \frac{e^{-1.3333}(1.3333)^0}{0!} = e^{-1.3333}$$

$$\boxed{P(X=0) = 0.2636}$$

(ii) $$P(X=3) = \frac{e^{-1.3333}(1.3333)^3}{3!}$$

$$= \frac{0.2636 \times 2.370}{6}$$

$$\boxed{P(X=3) = 0.1041}$$

3) Let X be a r.v following P.D such that $P(X=2) = 9P(X=4) + 90P(X=6)$. Find the mean and S.D of X

Soln:

W.K.T, $$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,2,\dots \quad \lambda > 0$$

Given: $P(X=2) = 9P(X=4) + 90P(X=6)$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\left(\frac{e^{-\lambda}\lambda^4}{4!}\right) + 90\left(\frac{e^{-\lambda}\lambda^6}{6!}\right)$$

$$\frac{e^{-\lambda}\lambda^2}{2} = \frac{9}{24} e^{-\lambda}\lambda^4 + \frac{90}{720} e^{-\lambda}\lambda^6$$

$$\frac{e^{-\lambda}\lambda^2}{2} = \frac{9}{24} e^{-\lambda}\lambda^4 \left[1 + \frac{10}{30}\lambda^2\right]$$

$$1 = \frac{9}{12}\lambda^2\left(\frac{3+\lambda^2}{3}\right)$$

$$4 = 3\lambda^2 + \lambda^4$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

Put $\lambda^2 = x$

$$x^2 + 3x - 4 = 0$$

$$x = 1, -4$$

$\lambda^2 = 1$ & $\lambda^2 = -4 \Rightarrow \boxed{\lambda = \pm 2i}$

$\Rightarrow \boxed{\lambda = \pm 1}$

$$\boxed{\lambda = 1, -1, 2i, -2i}$$

$\therefore \boxed{\lambda = 1}$ // ($\because$ λ must be real & positive)

Mean = variance = $\lambda = 1$

S.D = $\sqrt{\lambda} = \sqrt{1} = 1$

$\therefore$ $\boxed{\text{Mean} = \text{S.D} = 1}$

4) If X is a Poisson variate with mean λ, show that $E(X^2) = \lambda E(X+1)$.

Soln:

W.K.T, $E(X^2) = \lambda^2 + \lambda \longrightarrow (1)$

$E(X+1) = E(X) + 1$

$E(X+1) = \lambda + 1 \longrightarrow (2)$

L.H.S

$E(X^2) = \lambda^2 + \lambda$

$= \lambda(\lambda + 1)$ (By (1))

R.H.S

$\lambda E(X+1) = \lambda(\lambda+1) \longrightarrow$ (By (2))

L.H.S = R.H.S

Hence proved.

5) If for a Poisson variate X, $E(X^2) = 6$, what is $E(X)$?

Soln:

W.K.T, $E(X^2) = \lambda^2 + \lambda = 6$

$\Rightarrow \lambda^2 + \lambda - 6 = 0$

$\Rightarrow \lambda = -3, 2$

$\lambda = 2$ ($\because \lambda$ must be +ve)

$$\boxed{E(X) = \lambda = 2}$$

6) If X is a Poisson variate such that $P(X=2) = \frac{2}{3} P(X=1)$. Evaluate $P(X=3)$.

Soln:

Given; $P(X=2) = \frac{2}{3} P(X=1)$.

W.K.T: $P(X=x) = \frac{e^{-x} \lambda^{x}}{x!}$

Now, $P(X=2) = \frac{2}{3} P(X=1)$

$\Rightarrow 3P(X=2) = 2P(X=1)$

$\Rightarrow \frac{3 \cdot e^{-\lambda} \lambda^{2}}{2!} = \frac{2 e^{-\lambda} \lambda^{1}}{1!}$

$\Rightarrow 3e^{-\lambda} \lambda^{2} = 4e^{-\lambda} \lambda$

$\Rightarrow 3\lambda = 4$

$\Rightarrow \lambda = 4/3$

$\Rightarrow \boxed{\lambda = 1.333}$

$P(X=3) = \frac{e^{-1.333} (1.333)^{3}}{3!}$

$\boxed{P(X=3) = 0.1041}$

7) The M.G.F of a random variable X is given by $M_X(t) = e^{3(e^{t}-1)}$. Find $P(X=1)$.

Soln:

W.K.T, $M_X(t) = e^{\lambda(e^{t}-1)} \longrightarrow (1)$

Given, $M_X(t) = e^{3(e^{t}-1)} \longrightarrow (2)$

Comparing (1) and (2),

$$\boxed{\lambda = 3}$$

$$P(X=1) = \frac{e^{-3}(3)^1}{1!}$$

$$= 3e^{-3}$$

$$\boxed{P(X=1) = 0.1494}$$

8) Let X be a r.v which assumes P.D.

(i) Find $P(X=4)$ if $P(X=1) = P(X=2)$

(ii) Find $E(X)$ if $2P(X=0) + P(X=2) = 2P(X=1)$.

<u>Soln:</u>

(i) Given: $P(X=1) = P(X=2)$

$$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$$

$$\boxed{\lambda = 2}$$

then, $P(X=4) = \dfrac{e^{-\lambda}\lambda^4}{4!} = \dfrac{e^{-2}(2)^4}{1\times2\times3\times4}$

$$= \frac{e^{-2}\times16}{2\times3\times4} = \frac{2e^{-2}}{3}$$

$$\boxed{P(X=4) = 0.0902}$$

(ii) Given: $2P(X=0) + P(X=2) = 2P(X=1)$

$$\frac{2e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^2}{2!} = \frac{2 \cdot e^{-\lambda}\lambda^1}{1!}$$

$$2e^{-\lambda} + \frac{e^{-\lambda}\lambda^2}{2} = 2e^{-\lambda} \cdot \lambda$$

$$4e^{-\lambda} + e^{-\lambda}\lambda^2 = 4e^{-\lambda} \cdot \lambda$$

$$e^{-\lambda}(4+\lambda^2) = 4e^{-\lambda} \cdot \lambda$$

$$4 + \lambda^2 - 4\lambda = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda-2)(\lambda-2) = 0$$

$$\lambda = 2, 2$$

$$\boxed{E(X) = \lambda = 2}$$

9) If a Poisson variate X is such that $P(X=1) = 2P(X=2)$. Find $P(X=0)$ and $\text{var}(X)$.

Soln:

Given: $P(X=1) = 2P(X=2)$

$$\frac{e^{-\lambda}\lambda}{1!} = \frac{2e^{-\lambda}\lambda^2}{2!}$$

$$\boxed{\lambda = 1}$$

Now, $P(X=0) = \dfrac{e^{-1}(1)^0}{0!} = e^{-1} = 0.3679$

$$\boxed{\text{Var}(X) = \lambda = 1}$$

Note:

Examples of R.V's that usually obey the Poisson Distribution:

(i) The no. of Misprints on a page of a book.

(ii) The no. of people in a community living upto 100 years of age.

(iii) The no. of wrong calls among telephone numbers that are dialled in a day.

(iv) The no. of accidents in some unit of time.

10) In a company the monthly break down of a machine is a random variable with Poisson distribution, with an average 1.8. Find the probability that the machine will function for a month.

(i) without break down,

(ii) with exactly one break down,

(iii) with at least one break down.

Soln:

Let X denotes the number of break downs in a month.

Now, $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

Given: Mean $= \lambda = 1.8$

$$P(X=x) = \frac{e^{-1.8}(1.8)^{x}}{x!}$$

(i) P(without break down) = $P(X=0)$

$$= \frac{e^{-1.8}(1.8)^0}{0!}$$

$$\boxed{P(X=0) = 0.1653}$$

(ii) P(with exactly one breakdown) = $P(X=1)$

$$= \frac{e^{-1.8}(1.8)^1}{1!}$$

$$= 1.8\, e^{-1.8}$$

$$\boxed{P(X=1) = 0.2975}$$

(iii) P(with atleast one breakdown) = $P(X \geq 1)$

$$= 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - 0.1653$$

$$\boxed{P(X \geq 1) = 0.8347}$$

11) A book of 500 pages contains 500 mistakes. Find the probability that there are at least four mistakes in a randomly selected page.

Soln:

Total number of mistakes = 500

Total number of pages = 500

There is an average of 1 mistake per page

i.e., $\boxed{\lambda = 1}$

Let X be a random Variable of number of mistakes in a page.

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-1}(1)^{x}}{x!}$$

$$P(\text{at least four mistakes}) = P(X \geq 4)$$

$$= 1 - P(X < 4)$$

$$= 1 - P(X \leq 3)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$= 1 - \left[\frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!} + \frac{e^{-1}(1)^3}{3!}\right]$$

$$= 1 - \left[e^{-1}\left[1 + 1 + \frac{1}{2} + \frac{1}{6}\right]\right]$$

$$= 1 - \left[0.3679\left(2 + \frac{1}{2} + \frac{1}{6}\right)\right]$$

$$= 1 - [0.3679(2.6667)]$$

$$= 1 - 0.9811$$

$$= 0.0189$$

12) A manufacturer of wet grinders wants to buy one-hp motors from a Supplier, in a lot of 1000. When fitted to the machine, these motors have the probability of failure 0.001. In a Shipment of 1000 motors what is the Probability

that (i) none are defective,

(ii) One is defective,

(iii) Two are defective,

(iv) Three are defective.

<u>Soln</u>:

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

<u>Given</u>: $n=1000$, $p=0.001$

$$\Rightarrow \lambda = np$$

$$= 1000 \times 0.001$$

$\boxed{\lambda = 1}$ This distribution is P.D.

W.K.T, $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

(i) $P(\text{no defective}) = P(X=0) = \dfrac{e^{-1}1^{0}}{0!} = 0.3679$

(ii) $P(\text{one is defective}) = P(X=1) = \dfrac{e^{-1}(1)^{1}}{1!} = 0.3679$

(iii) $P(\text{two is defective}) = P(X=2) = \dfrac{e^{-2}(1)^{2}}{2!} = 0.1839$

(iv) $P(\text{three is defective}) = P(X=3) = \dfrac{e^{-1}(1)^{3}}{3!} = 0.0613$

13) In a book of 520 pages, 390 typographical errors occur. Assuming X is a Poisson variate for number of errors per page, find the probability

that a random Sample of 5 pages will contain no error.

Soln:

The average number of typographical errors per page $= \lambda = \frac{390}{520} = 0.75$

$\therefore \boxed{\lambda = 0.75}$

Now, X be a poisson variate

w.k.T, $P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$

P(1 page contain no error) $= P(X=0)$

$= \frac{e^{-0.75}(0.7)^{0}}{0!} = e^{-0.75}$

$= 0.4724$ //

P(5 pages contain no error) $= [P(X=0)]^{5} = (e^{-0.75})^{5}$

$= (0.4724)^{5}$

$= 0.0235$ //

14) In a component manufacturing industry, there is a Small probability of $\frac{1}{500}$ for any component to be defective. The components are supplied in packets of 10. Use poisson distribution to Calculate the approximate number of packets

containing (i) no defective,

(ii) one defective component,

(iii) two defective components in a consignment of 10,000 packets.

Soln:

Given $p = \frac{1}{500}$; $n = 10$

Let X be the no. of defectives in a packet.

W.K.T, $\lambda = np$.

$= 10 \cdot \frac{1}{500}$

$\lambda = \frac{1}{50}$

$\boxed{\lambda = 0.02}$

W.K.T, $P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$

(i) $P(\text{no defective}) = P(X=0)$

$= \frac{e^{-0.02} (0.02)^{0}}{0!}$

$= 0.9802$

The no. of packets free from defective is

$= 10,000 \times 0.9802$

$= 9802$ packets //

(ii) P(one defective) = $P(X=1)$

$$= \frac{e^{-0.02}(0.02)^1}{1!}$$

$$= 0.9802 \times 0.02$$

$$= 0.0196$$

The no. of packets with one defective } $= 10{,}000 \times 0.0196$

$= 196$ packets //

(iii) P(two defective) = $P(X=2)$

$$= \frac{e^{-0.02}(0.02)^2}{2}$$

$$= \frac{0.9082 \times 0.004}{2}$$

$$= 0.0002$$

The no. of packets with two defective } $= 10{,}000 \times 0.0002$

$= 2$ packets //

15) It is known that the probability that an item produced by a certain machine will be defective is 0.05. If the produced item was sent to the market in packets of 20,

find the number of packets containing at least, exactly and atmost 2 defective item in a consignment of 1000 packets using Poisson approximation. 4.42

Soln:

Given : $n = 20$, $p = 0.05$

$\lambda = np$

$= 20 \times 0.05$

$\boxed{\lambda = 1}$

Let X be a Poisson Variate.

w.k.T, $P(X = x) = \dfrac{e^{-\lambda} \lambda^x}{x!}$

(i) $P(\text{atleast 2 defective}) = P(X \geq 2)$

$= 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[\dfrac{e^{-1}(1)^0}{0!} + \dfrac{e^{-1}(1)^1}{1!}\right]$

$= 1 - [0.3679 + 0.3679]$

$= 1 - 0.7358$

$= 0.2642$ //

$\therefore$ The no. of packets containing at least 2 defectives $\Big\} = 0.2642 \times 1000$

$\simeq \boxed{264}$ packets

(ii) P(exactly 2 defective) = $P(X=2)$

$$= \frac{e^{-1}(1)^2}{2!}$$

$$= \frac{0.3679}{2}$$

$$= 0.1840$$

The no. of packets containing exactly 2 defective $\Big\}$ $= 0.1840 \times 1000$

$\simeq \boxed{184}$ packets //

(iii) P(atmost 2 defective) $= P(X \leq 2)$

$$= P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!}$$

$$= 0.3679 + 0.3679 + 0.1840$$

$$= 0.9198$$

The no. of packets containing atmost 2 defective $\Big\}$ $= 0.9198 \times 1000$

$= 919.8$

$\simeq \boxed{920}$ packets //

Geometric Distribution:

Characteristics of Geometric distribution:

(i) The trials Bernoulli's trials with Success (s) and failure (f).

(ii) The trials are identical and independent. The probability of success must remain the same from trial to trial.

(iii) The random variable X denotes the number of trials needed to obtain the first success.

Note: The sample space for an experiment can be described as

$$S = \{s, fs, ffs, fffs, ffffs, \ldots\}$$

Definition:

A random variable X is said to have a geometric distribution with parameter p if the probability mass function is given by

$$P(X = x) = q^{x-1} p \, ; \qquad x = 1, 2, 3, \ldots$$

where, $q = 1 - p$

M.G.F of Geometric distribution :

W.K.T, $M_X(t) = E(e^{tX})$

$$= \Sigma e^{tx} P(X=x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= p\left[e^{t}q^{0} + e^{2t}q^{1} + e^{3t}q^{2} + \cdots\right]$$

$$= pe^{t}\left[q^{0} + qe^{t} + q^{2}e^{2t} + \cdots\right]$$

$$= pe^{t}\left[1 + qe^{t} + (qe^{t})^{2} + \cdots\right]$$

$$= pe^{t}(1 - qe^{t})^{-1}$$

$$\boxed{M_X(t) = \frac{pe^{t}}{1-qe^{t}}}$$

Mean:

$$\mu_1' = [M_X'(t)]_{t=0} = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{pe^{t}}{1-qe^{t}}\right)\right]_{t=0}$$

$$= \left[\frac{(1-qe^{t})pe^{t} - pe^{t}(-qe^{t})}{(1-qe^{t})^{2}}\right]_{t=0}$$

$$= \left[\frac{pe^{t}(1-qe^{t}+qe^{t})}{(1-qe^{t})^{2}}\right]_{t=0}$$

$$= \left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\boxed{\mu_1' = \text{Mean} = \frac{1}{p}}$$

$$\mu_2' = \left\{\frac{d^2}{dt^2}\left[M_X(t)\right]\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]\right\}_{t=0}$$

$$= \left[\frac{(1-qe^t)^2 pe^t - pe^t\left[2(1-qe^t)(-qe^t)\right]}{(1-qe^t)^4}\right]_{t=0}$$

$$= \frac{(1-q)^2 p - p\left[2(1-q)(-q)\right]}{(1-q)^4}$$

$$= \frac{p^2 p + 2p^2 q}{p^4}$$

$$= \frac{p^2(p+2q)}{p^4}$$

$$\boxed{\mu_2' = \frac{p+2q}{p^2}}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= \frac{p+2q}{p^2} - \frac{1}{p^2}$$

$$= \frac{p+2q-1}{p^2}$$

$$= \frac{2q-q}{p^2} = \frac{q}{p^2}$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

Another form of geometric distribution:

If X denotes the number of failures before the first success, then

$$\boxed{P(X=x) = q^x p, \quad x = 0, 1, 2, 3, \ldots}$$

In this case,

* $\boxed{\text{M.G.F} = \frac{p}{1-qe^t}}$

* $\boxed{\text{Mean} = \frac{q}{p}}$

* $\boxed{\text{Variance} = \frac{q}{p^2}}$

Memoryless property of geometric distribution:

If X is a random variable with geometric distribution, then X lacks memory, in the sense that $P[X>s+t / X>s] = P[X>t]$

Proof:

$$P[X>s+t / X>s] = \frac{P[X>s+t \cap X>s]}{P[X>s]}$$

$$= \frac{P[X>s+t]}{P[X>s]} \longrightarrow (I)$$

Now $P[X=x] = q^{x-1}p, \quad x = 1, 2, 3, \ldots$

Now $P[X>k] = \sum_{x=k+1}^{\infty} q^{x-1} p$

$$= q^k p + q^{k+1} p + q^{k+2} p + \ldots$$

$$= q^k p \left[1 + q + q^2 + \ldots\right]$$

$$= q^k p \left[1-q\right]^{-1}$$

$$P[X>k] = \frac{q^k p}{1-q}$$

$$= \frac{q^k p}{p}$$

$$\boxed{P[X>k] = q^k} \quad (X)$$

Hence $P[X > s+t] = q^{s+t}$

$P[X > s] = q^{s}$

$$(I) \Rightarrow P[X > s+t \mid X > s] = \frac{q^{s+t}}{q^{s}} = \frac{q^{s} \cdot q^{t}}{q^{s}} = q^{t} \rightarrow (1)$$

Also, $P[X > t] = q^{t} \rightarrow (2)$

From (1) & (2),

$$P[X > s+t \mid X > s] = P(X > t)$$

Hence the geometric distribution lacks memory.

Problems:

1) A die is tossed until 6 appears. What is the probability that it must be tossed more than 5 times?

Soln:

Let X be the no. of tosses required to get the first 6.

The probability of getting $6 = \frac{1}{6}$

$$\therefore \boxed{p = 1/6}$$

$$\Rightarrow q = 1 - 1/6 = 5/6$$

$$\boxed{q = 5/6}$$

4.50

The p.m.f of G.D is

$$P(X=x) = q^{x-1} \cdot p \quad , \quad x = 1,2,3,\ldots$$

$$P(X>5) = 1 - P(X \leq 5)$$

$$= 1 - [P(X=1) + \cdots + P(X=5)]$$

$$= 1 - [q^0 p + q^1 p + q^2 p + q^3 p + q^4 p]$$

$$= 1 - [p + qp + q^2 p + q^3 p + q^4 p]$$

$$= 1 - \left[\frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + (5/6)^2 (1/6) + (5/6)^3 (1/6) + (5/6)^4 (1/6)\right]$$

$$= 1 - \left[\frac{1}{6} + \frac{5}{36} + \frac{25}{36 \times 6} + \frac{125}{36 \times 36} + \frac{625}{36 \times 36 \times 6}\right]$$

$$= 1 - \left[\frac{1}{6} + \frac{5}{36} + \frac{25}{216} + \frac{125}{1296} + \frac{625}{7776}\right]$$

$$= 1 - \left[\frac{1296 + 1080 + 900 + 750 + 625}{7776}\right]$$

$$= 1 - \frac{4651}{7776}$$

$$= 1 - 0.5981$$

$$\boxed{P(X>5) = 0.4019}$$

2) If X is G.D taking the values $1, 2, \ldots, \infty$. Find $P(X \text{ is odd})$.

Soln:

p.m.f of G.D is $P(X=x) = q^{x-1} p, \ (x = 1, 2, \ldots)$

$$P(X \text{ is odd}) = \sum_{x=1,3,5,\dots}^{\infty} q^{x-1} p \quad (\because x \text{ is odd})$$

$$= q^0 p + q^2 p + q^4 p + \cdots$$

$$= p + q^2 p + q^4 p + \cdots$$

$$= p[1 + q^2 + q^4 + \cdots]$$

$$= p[1 + (q^2)^1 + (q^2)^2 + \cdots]$$

$$= p(1-q^2)^{-1}$$

$$= \frac{p}{1-q^2}$$

$$= \frac{p}{(1-q)(1+q)}$$

$$= \frac{p}{p(1+q)}$$

$$P(X \text{ is odd}) = \frac{1}{1+q}$$ //

3) If the probability of success on each trial is $\frac{1}{3}$, what is the expected number of trials required for the first success.

Soln:

Let X be the no. of trials required for the first success, then X follows G.D with

$$P(X=x) = q^{x-1} p, \quad (x = 1, 2, 3, \dots)$$

Here, Mean $= \frac{1}{p}$

$= \frac{1}{1/3}$

$\boxed{\text{Mean} = 3}$

4) If $M_X(t) = (5-4e^t)^{-1}$. Find $P(X=5 \text{ or } 6)$

Soln:

Given $M_X(t) = (5-4e^t)^{-1}$

$= \frac{1}{5-4e^t}$

$= \frac{1}{5(1-4/5\, e^t)}$

$M_X(t) = \frac{1/5}{\left(1-\frac{4}{5}e^t\right)} \rightarrow (1)$

W.K.T,

$M_X(t) = \frac{p}{(1-qe^t)}$, $x = 0, 1, 2, \ldots$ $\rightarrow (2)$

Comparing (1) & (2), we get

$p = 1/5$, $q = 4/5$

To find: $P(X=5 \text{ or } 6)$

W.K.T, $P(X=x) = q^x p$, $x = 0, 1, 2, \ldots$

$$P(X=5 \text{ or } 6) = P(X=5) + P(X=6)$$
$$= q^5 p + q^6 p$$
$$= \left(\frac{4}{5}\right)^5 \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^6 \left(\frac{1}{5}\right)$$
$$= \frac{4^5}{5^6} + \frac{4^6}{5^7}$$
$$= \frac{5120 + 4096}{78125}$$
$$= \frac{9216}{78125}$$
$$P(X=5 \text{ or } 6) = 0.11796$$

5) The probability that a certain measuring device will show excessive drift, is 0.10. What is the probability that the fifth of these measuring devices tested will be the first to show excessive drift. Find its expected value also.

Soln:

$\boxed{p=0.10}$, $q = 1-0.10 = \boxed{0.90} \Rightarrow \boxed{q=0.90}$

$$P(X=x) = q^{x-1} p, \quad x = 1, 2, \ldots$$
$$P(X=5) = (0.90)^{5-1}(0.10)$$
$$= (0.90)^4 (0.10)$$
$$\boxed{P(X=5) = 0.0656}$$

$$E(X) = \frac{1}{p}$$

$$= \frac{1}{0.10}$$

$$\boxed{E(X) = 10}$$

6) Identify the distribution with MGF,

$$M_X(t) = e^t (5-4e^t)^{-1}$$

Soln:

Given $M_X(t) = e^t (5-4e^t)^{-1}$

$$= \frac{e^t}{5-4e^t}$$

$$= \frac{e^t}{5(1-4/5\, e^t)}$$

$$M_X(t) = \frac{1/5\, e^t}{(1-4/5\, e^t)} \longrightarrow (1)$$

W.K.T,

$$M_X(t) = \frac{pe^t}{(1-qe^t)}, \; x=1,2,\ldots \longrightarrow (2)$$

Comparing (1) and (2), we get

$$p = \frac{1}{5}, \quad q = \frac{4}{5}$$

Hence, this is the G.D and the parameter $(p) = 1/5$ //.

H.W

14) In a distribution exactly normal, 7% of items are under 35 and 89% are under 63. What are the mean and S.D of the distribution.

Soln:

Let Mean $= \mu$, S.D $= \sigma$

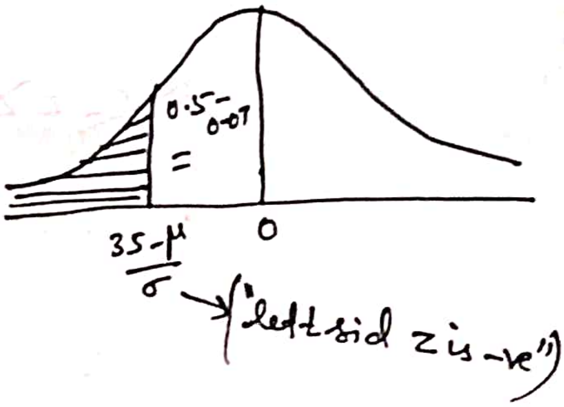
W.K.T, $z = \dfrac{x-\mu}{\sigma}$

Here, 7% of the items are under 35.

i.e., $P(X < 35) = 0.07$

$P\left(\dfrac{x-\mu}{\sigma} < \dfrac{35-\mu}{\sigma}\right) = 0.07$

$P\left(z < \dfrac{35-\mu}{\sigma}\right) = 0.07$



$0.5 - P\left(\boxed{\dfrac{35-\mu}{\sigma}} < z < 0\right) = 0.07$ (left side region)

$0.5 - P\left(0 < z < \dfrac{35-\mu}{\sigma}\right) = 0.07$

Note:
35% region
⇓
"z" must be left hand side

$P\left(0 < z < \dfrac{35-\mu}{\sigma}\right) = 0.5 - 0.07$

$P\left(0 < z < \dfrac{35-\mu}{\sigma}\right) = 0.43$

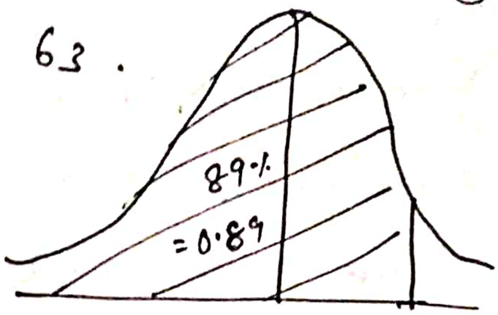
$z = \dfrac{35-\mu}{\sigma} = 1.48$

$\dfrac{35-\mu}{\sigma} = -1.48$ (left side region) ("z" value must be "-ve")

$\boxed{\mu - 1.48\sigma - 35 = 0} \rightarrow (1)$

Here 89% of the items are under 63.

(ie) $P(X < 63) = 0.89$



$$P\left(\frac{X-\mu}{\sigma} < \frac{63-\mu}{\sigma}\right) = 0.89$$

$$P\left(Z < \frac{63-\mu}{\sigma}\right) = 0.89$$

↓

(right side z is +ve)

$$0.5 + P\left(0 < Z < \frac{63-\mu}{\sigma}\right) = 0.89$$

$$P\left(0 < Z < \frac{63-\mu}{\sigma}\right) = 0.89 - 0.5$$

Note: 89% region ⇓ "z" must be right hand side

$$P\left(0 < Z < \frac{63-\mu}{\sigma}\right) = 0.39 \quad \text{(Right side region)}$$

Hence $Z = \frac{63-\mu}{\sigma} = 1.23$ ('z' value must be +ve)

$$\boxed{\mu + 1.23\sigma - 63 = 0} \rightarrow (2)$$

Solve (1) & (2), we get

$$\boxed{\mu = 50.29 \text{ and } \sigma = 10.33}$$

Unit - 5

Theoretical continuous Distribution:

Exponential Distribution:

A Continuous random variable X defined in $(0,\infty)$ is said to follow an exponential distribution if the probability density function is

$$\boxed{f(x) = \lambda e^{-\lambda x}, \quad \lambda > 0, \quad 0 < x < \infty}$$

where λ is the parameter.

Cumulative distribution function:

$$\boxed{F(x) = \begin{cases} \int_0^x \lambda e^{-\lambda x}\,dx & : \text{if } x > 0 \\ 0 & : \text{otherwise} \end{cases}}$$

(or)

$$\boxed{F(x) = \begin{cases} 1 - e^{-\lambda x} & : x > 0 \\ 0 & : \text{otherwise} \end{cases}}$$

The r^{th} moment is given by

$$\boxed{\mu_r' = E(x^r) = \frac{r!}{\lambda^r}}$$

$$\boxed{\text{Mean } (\mu_1') = \frac{1}{\lambda}}$$

$$\mu_2' = \frac{2}{\lambda^2}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

Moment generating function, Mean and Variance :

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\,dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty = \frac{\lambda}{\lambda - t}$$

$$\boxed{M_X(t) = \frac{\lambda}{\lambda - t}}$$

Mean:

$$\mu_1' = \left[\frac{d}{dt}\left(M_X(t)\right) \right]_{t=0}$$

$$= \left\{ \frac{d}{dt} \left[\frac{\lambda}{\lambda - t} \right] \right\}_{t=0}$$

$$= \left\{ \frac{(\lambda - t)\cdot 0 - \lambda(-1)}{(\lambda - t)^2} \right\}_{t=0}$$

$$= \left[\frac{\lambda}{(\lambda - t)^2} \right]_{t=0}$$

$$= \frac{\lambda}{\lambda^2}$$

$$\boxed{\mu_1' = 1/\lambda}$$

Variance:

$$\mu_2' = \left\{ \frac{d^2}{dt^2} \left[M_X(t) \right] \right\}_{t=0}$$

$$= \left[\frac{d}{dt} \left(\frac{\lambda}{(\lambda - t)^2} \right) \right]_{t=0}$$

$$= \left[\frac{-2\lambda}{(\lambda - t)^2} (-1) \right]_{t=0}$$

$$= \left[\frac{2\lambda}{(\lambda - t)^2} \right]_{t=0}$$

$$= \frac{2\lambda}{\lambda^3}$$

$$\boxed{\mu_2' = \frac{2}{\lambda^2}}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

Memoryless property of exponential distribution:

If X is exponentially distributed with parameter λ, then for any two positive integers 's' and 't',

$$P[X > s+t / X > s] = P[x > t]$$

Proof:

The p.d.f of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; \; x > 0 \\ 0 & ; \; \text{otherwise} \end{cases}$$

Now $P[X > k] = \int_k^\infty \lambda e^{-\lambda x}$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_k^\infty$$

$$\boxed{P[X > k] = e^{-\lambda k}}$$

Consider for $s, t \geq 0$,

$$P[X > s+t / X > s] = \frac{P[X > s+t \cap X > s]}{P[X > s]}$$

$$= \frac{P[X > s+t]}{P[X > s]}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = \frac{e^{-\lambda s} \cdot e^{-\lambda t}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$= P(X > t)$$

Thus $P[X > s+t / X > s] = P[X > t] \quad \forall\, s, t \geq 0$

Problems:

1) Let X be a r.v with exponentially distributed, with parameter λ. Find variance of X, if $P(X \leq 1) = P(X > 1)$

Soln:

Given: $\boxed{P(X \leq 1) = P(X > 1)} \longrightarrow (1)$

$$P(X \leq 1) = 1 - P(X \leq 1)$$

$$2P(X \leq 1) = 1 \quad \Rightarrow \quad \boxed{P(X \leq 1) = \frac{1}{2}} \longrightarrow (2)$$

Comparing (1) and (2)

$$P(X>1) = \frac{1}{2}$$

$$e^{-\lambda} = \frac{1}{2} \quad \left(\because P(X>t) = e^{-\lambda t} \text{ Here } t=1 \right)$$

$$\frac{1}{e^{\lambda}} = \frac{1}{2}$$

$$\boxed{e^{\lambda} = 2}$$

Taking log on both sides

$$\log e^{\lambda} = \log 2$$

$$\lambda = \log 2$$

$$Var(X) = \frac{1}{\lambda^2}$$

$$\boxed{Var(X) = \frac{1}{(\log 2)^2}}$$

2) The number of kilometers that a car can run before its battery has to be replaced is exponentially distributed with an average of 10,000 kms. If the owner desires to take a tour consisting of 5000 kms, what is the probability that he will be able to complete his tour without replacing the battery.

Soln:

57.

Let X be a random variable representing the number of Kilometers that the car can run without replacing the battery.

Given X is exponentially distributed.

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

Also mean = 10,000

$$\Rightarrow \lambda = \frac{1}{10,000}$$

To find,

$$P(X > 8000) = \int_{8000}^{\infty} f(x)\,dx$$

$$= \int_{8000}^{\infty} \frac{1}{10,000} e^{-\frac{x}{10,000}}\,dx$$

$$= \frac{1}{10,000}\left[\frac{e^{-x/10,000}}{-1/10,000}\right]_{8000}^{\infty}$$

$$= -\left[0 - e^{-8/10}\right]$$

$$= e^{-0.8}$$

$$\boxed{P[X > 8000] = 0.4493}$$

3) The length of time (in minutes) that a certain lady speaks on the telephone is a random variable specified by the p.d.f. (5.8.)

$$f(x) = \begin{cases} Ae^{-x/5} & : x>0 \\ 0 & : \text{otherwise} \end{cases}$$

Evaluate A. Find the probability that the number of minutes she talks on the phone is

(i) More than 10 minutes

(ii) Less than 5 minutes

(iii) Between 5 and 10 minutes

Soln:

Given $f(x) = Ae^{-x/5}, \quad x \geq 0$

Now, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{\infty} Ae^{-x/5}\,dx = 1$$

$$A\left[\frac{e^{-x/5}}{-1/5}\right]_0^{\infty} = 1$$

$$-5A\left[0 - e^{0}\right] = 1$$

$$5A = 1$$

$$\boxed{A = 1/5}$$

Hence $f(x) = \begin{cases} \frac{1}{5} e^{-x/5} & ; \ x \geq 0 \\ 0 & ; \ \text{otherwise} \end{cases}$

(i) $P(\text{more than 10 minutes}) = P(X > 10)$

$$= \int_{10}^{\infty} \frac{1}{5} e^{-x/5} \, dx$$

$$= \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_{10}^{\infty}$$

$$= -\left[0 - e^{-10/5} \right]$$

$$= e^{-2} = 0.1353$$

(ii) $P(\text{less than 5 minutes}) = P(X < 5)$

$$= \int_{0}^{5} \frac{1}{5} e^{-x/5} \, dx$$

$$= \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_{0}^{5}$$

$$= -\left[e^{-1} - 1 \right]$$

$$= 1 - 0.3679 = 0.6321$$

(iii) $P(\text{between 5 and 10 minutes}) = P(5 < X < 10)$

$$= \int_{5}^{10} \frac{1}{5} e^{-x/5} \, dx$$

$$= \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_{5}^{10}$$

$$= -\left[e^{-10/5} - e^{-5/5}\right]$$

$$= -\left[e^{-2} - e^{-1}\right]$$

$$= (e^{-1} - e^{-2})$$

$$= (0.3679 - 0.1353)$$

$$= 0.2326$$

4) The mileage which car owners get with a certain type of radial tyre is a random variable having an exponential distribution with mean 40,000 km. Find the probabilities that one of these tyres will last

(i) at least 20,000 km

(ii) atmost 30,000 km.

Soln:

Given: mean = 40,000 km

$$\Rightarrow \lambda = \frac{1}{\text{mean}} = \frac{1}{40,000}$$

now, $f(x) = \dfrac{1}{40,000} e^{-x/40000}$, $x > 0$

(i) P(atleast 20,000 kms) = $P(X \geq 20,000)$

$$= \int_{20000}^{\infty} \frac{1}{40,000} e^{-x/40,000} \, dx$$

$$= \frac{1}{40{,}000}\left[\frac{e^{-x/40000}}{-1/40{,}000}\right]_{20{,}000}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-\frac{20{,}000}{40{,}000}}\right]$$

$$= e^{-1/2} = 0.6065$$

(ii) P[atmost 30,000 kms]

$$= P[X \le 30{,}000] = \int_0^{30{,}000} \frac{1}{40{,}000} e^{-x/40{,}000}\, dx$$

$$= \frac{1}{40{,}000}\left[\frac{e^{-x/40{,}000}}{-1/40{,}000}\right]_0^{30{,}000}$$

$$= -\left[e^{-\frac{30000}{40{,}000}} - 1\right]$$

$$= \left[1 - e^{-3/4}\right]$$

$$= 0.5276$$

5) The amount of time that a watch can run without having to be reset is a random variable having exponential distribution, with mean 120 days. Find the probability that such a watch will.

(i) have to be reset in less than 24 days

(ii) not have to be reset for at least 180 days.

Soln:

Let X denote the number of days the watch will run without reset.

Then $f(x) = \lambda e^{-\lambda x}$; $x \geq 0$

$$\text{Mean} = \frac{1}{\lambda} = 120 \text{ days}$$

$$\lambda = \frac{1}{120}$$

$$f(x) = \frac{1}{120} e^{-x/120} \; : \; x \geq 0$$

(i) $$P(X < 24) = \int_0^{24} \frac{1}{120} e^{-x/120} \, dx$$

$$= \frac{1}{120} \left[\frac{e^{-x/120}}{-1/120} \right]_0^{24}$$

$$= -\left[e^{-24/120} - 1 \right]$$

$$= 1 - e^{-1/5}$$

$$= 0.1813 \,//$$

(ii) $$P(X \geq 180) = \int_{180}^{\infty} \frac{1}{120} e^{-x/120} \, dx$$

$$= \frac{1}{120} \left[\frac{e^{-x/120}}{-1/120} \right]_{180}^{\infty} = -\left[e^{-\infty} - e^{-3/2} \right]$$

$$= e^{-3/2} = 0.2231 \,//$$

6) Suppose that the life of an industrial lamp, in thousands of hours, is exponentially distributed with mean life of 3000 hours. (5.13)

Find the probability

(i) that the lamp will last more than mean life.

(ii) that the lamp will last between 2000 and 3000 hrs.

(iii) that the lamp will last another 1000 hours given that it has already lasted for 2500 hours.

Soln:

Let X be the r.v of the life of industrial lamp.

Here, X is E.D with $\lambda = \frac{1}{\text{mean}} = \frac{1}{3000}$

Now $f(x) = \lambda e^{-\lambda x}, \quad x > 0$

$$f(x) = \frac{1}{3000} e^{-x/3000}, \quad x > 0$$

(i) $P(X > 3000) = \int_{3000}^{\infty} \frac{1}{3000} e^{-x/3000} dx$

$$= \frac{1}{3000} \left[\frac{e^{-x/3000}}{-1/3000} \right]_{3000}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-3000/3000} \right]$$

$$= -\left[0 - e^{-1} \right] = e^{-1} = 0.3679$$

(ii) $P(2000 < X < 3000) = \int_{2000}^{3000} \frac{1}{3000} e^{-x/3000} \, dx$

$= \frac{1}{3000} \left[\frac{e^{-x/3000}}{-1/3000} \right]_{2000}^{3000}$

$= -\left[e^{-1} - e^{-2/3} \right]$

$= -\left[0.3679 - 0.5134 \right]$

$= -0.3679 + 0.5134$

$= 0.1455$ //

(iii) $P(X > s+t / X > s) = P(X > t)$ (By Memoryless Property)

$P(X > 3500 / X > 2500) = P(X > 1000)$ $\left(\because s+t = 3500, \ s = 2500 \Rightarrow t = 1000 \right)$

$= \int_{1000}^{\infty} \frac{1}{3000} e^{-x/3000} \, dx$

$= \frac{1}{3000} \left[\frac{e^{-x/3000}}{-1/3000} \right]_{1000}^{\infty}$

$= -\left[e^{-\infty} - e^{-1/3} \right]$

$= e^{-1/3}$

$= 0.7165$ //

7) If X is a variable with E.D and with Mean 2, find $P[X<1/X \not< 2]$

Soln:

Given: Mean = 2

$$\lambda = \frac{1}{2}$$

Now, $f(x) = \lambda e^{-\lambda x}, \quad x>0$

$$f(x) = \frac{1}{2} e^{-x/2}, \quad x>0$$

$$P[X<1 / X<2] = \frac{P[X<1 \cap X<2]}{P(X<2)}$$

$$= \frac{P(X<1)}{P(X<2)}$$

$$= \frac{\int_0^1 \frac{1}{2} e^{-x/2}\, dx}{\int_0^2 \frac{1}{2} e^{-x/2}\, dx}$$

$$= \frac{\left[\frac{e^{-x/2}}{-1/2}\right]_0^1}{\left[\frac{e^{-x/2}}{-1/2}\right]_0^2}$$

$$= \frac{e^{-1/2} - e^0}{e^{-1} - e^0}$$

$$= \frac{e^{-1/2} - 1}{e^{-1} - 1}$$

$$= \frac{0.6065 - 1}{0.3679 - 1}$$

$$= \frac{-0.3935}{-0.6321}$$

$$P(X<1/X<2) = 0.6225$$

8) The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$. What is the probability that the repair time exceeds 2 hours? What is the conditional Probability that the repair time takes at least 10 hours given that its duration exceeds 9 hours?

Soln: Let X be a r.v of time to repair the machine;

Given X is exponentially distributed with $\lambda = 1/2$

$$\therefore f(x) = \frac{1}{2} e^{-x/2}, \quad x > 0$$

(i) $$P(X>2) = \int_2^\infty \frac{1}{2} e^{-x/2}\, dx$$

$$= \frac{1}{2} \left[\frac{e^{-x/2}}{-1/2} \right]_2^\infty$$

$$= -[0 - e^{-1}]$$

$$= e^{-1} = 0.3679$$

(ii) $P(X>10 / X>9) = P(X>1)$ (By memoryless Property)

$$= \int_1^{\infty} \frac{1}{2} e^{-x/2}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_1^{\infty}$$

$$= e^{-1/2} = 0.6065$$

9) Find the M.G.F and r^{th} moment for the distribution whose p.d.f is

$$f(x) = ke^{-x} : 0 \le x < \infty$$

Find also the standard deviation.

Soln:

Given: p.d.f $f(x) = ke^{-x}$; $0 \le x < \infty$

$$\int_0^{\infty} f(x)\,dx = 1$$

$$\int_0^{\infty} ke^{-x}\,dx = 1$$

$$\Rightarrow k\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow \boxed{k=1}$$

$$\therefore \boxed{f(x) = e^{-x} ;\ 0 \le x < \infty}$$

(i) M.G.F :

$$M_X(t) = E[e^{tX}]$$

$$= \int_0^\infty e^{tx} e^{-x}\, dx$$

$$= \int_0^\infty e^{-(1-t)x}\, dx$$

$$= \left[\frac{e^{-(1-t)}}{-(1-t)}\right]_0^\infty = \frac{1}{1-t}, \quad (t<1)$$

$$= (1-t)^{-1} = 1 + t + t^2 + \cdots + t^r + \cdots \infty$$

W.K.T, $\mu_r' = \text{coeff.} \dfrac{t^r}{r!}$

$$M_X(t) = \frac{1}{0!} \times 0! + \frac{t}{1!} \times 1! + \frac{t^2}{2!} \times 2! + \cdots + \frac{t^r}{r!} \times r! + \cdots \infty$$

Now, $\boxed{\mu_r' = r!}$

$\therefore$ r^{th} moment is $r!$

$\mu_1' = 1! = 1$ & $\mu_2' = 2! = 2$

$$\text{Var} = \mu_2' - (\mu_1')^2$$

$$= 2 - 1 = 1 \Rightarrow \boxed{\text{Var} = 1}$$

$$S.D = \sqrt{\text{Var}} = \sqrt{1} = 1$$

$$\boxed{S.D = 1}$$

10) Let X be a r.v with p.d.f

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Find (i) $P(X>3)$; (ii) M.G.F of X; (iii) $E(X)$; (iv) var X.

Soln:

Given: $$f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

(i) $$P(X>3) = \int_3^\infty \frac{1}{3} e^{-x/3}\, dx$$

$$= e^{-1} = 0.3679$$

(ii) M.G.F of X

$$M_X(t) = E(e^{tX})$$

$$= \int_0^\infty e^{tx} \cdot \frac{1}{3} e^{-x/3}\, dx$$

$$= \frac{1}{3} \int_0^\infty e^{tx - x/3}\, dx$$

$$= \frac{1}{3} \int_0^\infty e^{-(1/3 - t)x}\, dx$$

$$= \frac{1}{3} \left[\frac{e^{-(1/3-t)x}}{-(1/3-t)} \right]_0^\infty$$

$$= \frac{-1}{3\left(\frac{1}{3}-t\right)} \left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{1-3t}[0-1]$$

$$\boxed{M_X(t) = \frac{1}{1-3t}}$$

(iii) $E(X)$:

$$M_X'(t) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{1}{1-3t}\right)\right]_{t=0}$$

$$= \left[\frac{(1-3t)(0) - 1(-3)}{(1-3t)^2}\right]_{t=0} = \left[\frac{3}{(1-3t)^2}\right]_{t=0}$$

$M_X'(t) = \text{Mean} = E(X) = 3$ //

(iv) variance:

$$M_X''(t) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{3}{(1-3t)^2}\right)\right]_{t=0}$$

$$= \left[\frac{(1-3t)^2(0) - 3\left[2(1-3t)\right](-3)}{(1-3t)^4}\right]_{t=0}$$

$$= \left[\frac{18(1-3t)}{(1-3t)^4}\right]_{t=0} \Rightarrow \boxed{\mu_2' = 18}$$

$$\text{var} = \mu_2' - (\mu_1')^2 = 18 - 9 = 9 \Rightarrow \boxed{\text{var} = 9}$$

Normal Distribution:

Definition:

A continuous r.v X with probability density function.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0$$

is said to follow normal distribution.

Here the parameters are σ and μ, where mean $= \mu$ and S.D $= \sigma$.

Note:

(i) The normal distribution is denoted by $X \sim N(\mu, \sigma)$ (or) $X \sim N(\mu, \sigma^2)$

(ii) This distribution is also called Gaussian distribution.

Characteristics of Normal Distribution and Normal Probability Curve:

(i) The curve is bell shaped and symmetrical about the line $x = \mu$.

(ii) Mean, Median and Mode of the distribution coincide.

(iii) x-axis is an asymptote to the curve.

(iv) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at the point $x=\mu$, given by $[p(x)]_{max} = \frac{1}{\sigma\sqrt{2\pi}}$.

(v) Since $f(x)$ is non-negative, the curve will not go below the x-axis.

Moment Generating Function of Normal Distribution:

By definition,

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\left[\text{Put } z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \quad dx = \sigma\, dz\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \cdot e^{-z^2/2}\, dz$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)}\, dz$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[z^2 - 2t\sigma z + \sigma^2 t^2 - \sigma^2 t^2\right]} dz$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\left[(z-\sigma t)^2 - \sigma^2 t^2\right]} dz \qquad \text{(5.23)}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 (z-\sigma t)^2} dz$$

$$\left[\begin{array}{l} \text{Put } z - \sigma t = u \\ \quad dz = du \end{array}\right]$$

$$= e^{\mu t + \frac{t^2\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

$$= e^{\mu t + \frac{t^2\sigma^2}{2}} \cdot \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2/2} du$$

$$\left[\begin{array}{l} \text{Put } y^2 = u^2/2 \\ y = u/\sqrt{2} \\ dy = \frac{du}{\sqrt{2}} \\ du = \sqrt{2}\, dy \end{array}\right]$$

$$= \frac{e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \cdot 2 \int_0^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \frac{2}{\sqrt{\pi}} e^{\mu t + \sigma^2 t^2/2} \int_0^{\infty} e^{-y^2} dy$$

$$= \frac{2}{\sqrt{\pi}} e^{\mu t + \sigma^2 t^2/2} \cdot \frac{\sqrt{\pi}}{2} \qquad \left(\because \int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}\right)$$

$$\boxed{M_X(t) = e^{\mu t + \sigma^2 t^2/2}} \quad //$$

Moments of Normal Distribution:

All odd order moments of a Normal distribution $N(\mu, \sigma^2)$ about its mean are zero, and its even order moments about the mean are given by the recurrence relation.

$$\boxed{\mu_{2n} = \sigma^2(2n-1)\,\mu_{2n-2}}$$

Mean and variance of Normal Distribution using M.G.F:

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)\right]_{t=0}$$

$$= e^0(\mu + 0) = \mu$$

$$\boxed{\text{Mean} = \mu}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)\right]\right\}_{t=0}$$

$$= \left[e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)^2 + e^{\mu t + \sigma^2 t^2/2}(\sigma^2)\right]_{t=0}$$

$$= \left[e^{0}(\mu)^2 + e^{0}\sigma^2\right]$$

$$\boxed{E(X^2) = \mu^2 + \sigma^2}$$

$$\text{variance}(X) = E(X^2) - [E(X)]^2$$

$$= \mu^2 + \sigma^2 - (\mu)^2$$

$$= \sigma^2$$

$$\boxed{Var(X) = \sigma^2}$$

Note:

Median and Mode of N.D is μ

$$\therefore \boxed{\text{Mean} = \text{Median} = \text{Mode} = \mu}.$$

Standard Normal Distribution:

If X is a random variable following normal distribution with parameter μ and σ, then $Z = \frac{X-\mu}{\sigma}$ is called the Standard normal variate and the p.d.f of the Standard Variate Z is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \; ; \quad -\infty < x < \infty$$

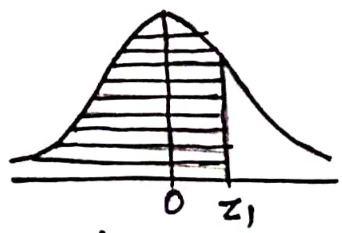
Basic Properties of Standard Normal Curve:

1) Total area under the standard normal curve is equal to 1.

2) The Standard normal curve is asymptotic to x-axis.

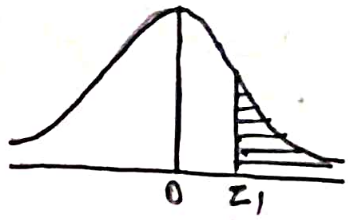
3) The standard normal curve is symmetric about 0.

4) Most of the area under the Standard normal curve lies b/w −3 and 3.



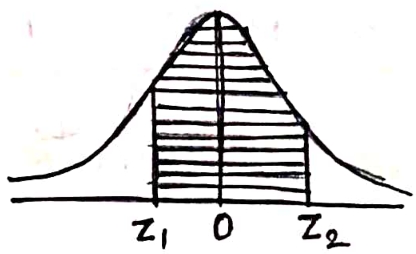
Shaded area $= P(Z < z_1)$

$= 0.5 + P(0 < Z < z_1)$



Shaded area $= P(Z > z_1)$

$= 0.5 - P(0 < Z < z_1)$



Shaded area $= P(z_1 < Z < z_2)$

$= P(0 < Z < z_1) + P(0 < Z < z_2)$

Additive property of Normal Distribution:

If $X_1, X_2, \ldots, X_n$ are independent normal variates with parameters $(\mu_1, \sigma_1), (\mu_2, \sigma_2), \ldots (\mu_n, \sigma_n)$ respectively, then $X_1 + X_2 + \cdots + X_n$ is also a normal variate with parameter (μ, σ). where $\mu = \mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$.

Proof:

The MGF of the random variable $X_i : i = 1, 2, 3, \ldots, n$ is $M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$, $i = 1, 2, \ldots, n$

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\cdot M_{X_2}(t)\cdots M_{X_n}(t)$$

[By property of MGF]

$$= e^{\mu_1 t+\frac{1}{2}\sigma_1^2 t^2}\cdot e^{\mu_2 t+\frac{1}{2}\sigma_2^2 t^2}\cdots e^{\mu_n t+\frac{1}{2}\sigma_n^2 t^2}$$

$$= e^{(\mu_1+\mu_2+\cdots+\mu_n)t+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)t^2}$$

$$= e^{\mu t+\frac{1}{2}\sigma^2 t^2}$$

which is the MGF of the variate $X_1+X_2+\cdots+X_n$.

So, $X_1+X_2+\cdots+X_n$ follows normal distribution with mean $\sum_{i=1}^{n}\mu_i$ and variance $\sum_{i=1}^{n}\sigma_i^2$.

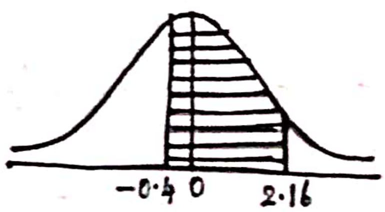
Problems:

1) Let X denote the number of grams of hydrocarbons emitted by an automobile per mile. Assuming X is normal with $\mu = 1$ gram and $\sigma = 0.25$, find the probability that a randomly selected automobile will emit between 0.9 and 1.54 gram of hydrocarbons per mile.

Soln:

To find: $P(0.9 \le X \le 1.54)$

Let $Z = \frac{X-\mu}{\sigma} = \frac{X-1}{0.25}$



$$P[0.9 \le X \le 1.54] = P\left[\frac{0.9-1}{0.25} \le \frac{X-1}{0.25} \le \frac{1.54-1}{0.25}\right]$$

$$= P[-0.4 \le Z \le 2.16]$$

$$= P(-0.4 \le Z \le 0) + P(0 \le Z \le 2.16)$$

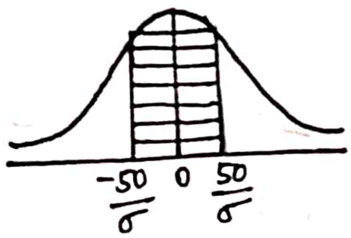
$= P(0 \leq z \leq 0.4) + P(0 \leq z \leq 2.16)$

$= 0.1554 + 0.4846$

$= 0.64$ //

2) The life time of an electric component is normally distributed with mean value of 250 hours and standard deviation of σ hours. Find the value of σ so that the probability of the component to have life between 200 and 300 hours is 0.7.

Soln:



Given: $\mu = 250$

$SD = \sigma$

$P(200 \leq X \leq 300) = 0.7$

$\Rightarrow P\left(\frac{200-250}{\sigma} \leq \frac{X-250}{\sigma} \leq \frac{300-250}{\sigma}\right) = 0.7$

$\Rightarrow P\left(\frac{-50}{\sigma} \leq z \leq \frac{50}{\sigma}\right) = 0.7$

$\Rightarrow 2P\left(0 \leq z \leq \frac{50}{\sigma}\right) = 0.7$

$\Rightarrow P\left(0 \leq z \leq \frac{50}{\sigma}\right) = 0.35$

$\therefore$ From the normal table,

$\frac{50}{\sigma} = 1.04$

$\therefore \sigma = \frac{50}{1.04} = 48.077$

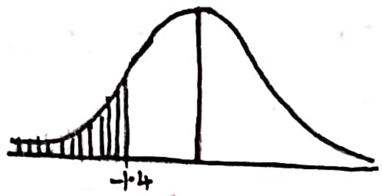
$\boxed{\sigma = 48.077}$ //

3) A certain type of Storage battery lasts on the average 3.0 years with Standard deviation of 0.5 year. Assuming that the battery lives are normally distributed, find the probability that a given battery will last less than 2.3 years.

Soln:

To find : $P(X < 2.3)$

Given $\mu = 3, \sigma = 0.5$



Now, $P(X < 2.3) = P\left(\frac{X-3}{0.5} < \frac{2.3-3}{0.5}\right)$

$= P\left(Z < \frac{2.3-3}{0.5}\right)$

$= P(Z < -1.4)$

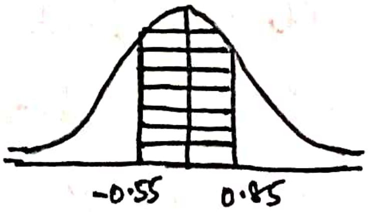
$= P(Z > 1.4)$

$= 0.5 - P(0 < Z < 1.4)$

$= 0.5 - 0.4192$

$= 0.0808$ //

4) An Electrical firm manufactures light bulbs that have a length of life which is normally distributed with $\mu = 800$ hrs and $\sigma = 40$ hrs. Find the probability that a bulb burns between 778 and 834 hours.



To find:

$$P(778 < X < 834) = P\left[\frac{778-800}{40} < \frac{X-800}{40} < \frac{834-800}{40}\right]$$

$$= P(-0.55 < z < 0.85)$$

$$= P(0 < z < 0.55) + P(0 < z < 0.85)$$

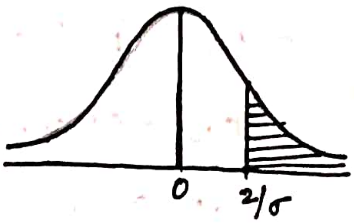
$$= 0.2088 + 0.3023$$

$$= 0.5111$$

5) Let X be a normally distributed random variable with mean $= 10$ and the probability $P[X > 12) = 0.1587$. What is the probability that X will be in the interval $(9, 11)$?

Soln:

Given: $\mu = 10$ & $P(X > 12) = 0.1587$



$$P(X > 12) = 0.1587$$

$$P\left[\frac{X-10}{\sigma} > \frac{12-10}{\sigma}\right] = 0.1587$$

$$P\left[z > \frac{2}{\sigma}\right] = 0.1587$$

$$0.5 - P\left[0 < z < \frac{2}{\sigma}\right] = 0.1587$$

$$P\left[0 < z < \frac{2}{\sigma}\right] = 0.5 - 0.1587$$

$$P\left(0 < z < \frac{2}{\sigma}\right) = 0.3413$$

$$\frac{2}{\sigma} = 1$$

$$\boxed{\sigma = 2}$$

To find: $P(9 < X < 11)$

$$P(9 < X < 11) = P\left[\frac{9-10}{2} < \frac{X-10}{2} < \frac{11-10}{2}\right]$$

$$= P\left[-\frac{1}{2} < z < \frac{1}{2}\right]$$

$$= 2P\left[0 < z < 1/2\right]$$

$$= 2P\left[0 < z < 0.5\right]$$

$$= 2 \times 0.1915$$

$$= 0.3830 //$$

6) Let X and Y be independent normal variates with means 45 and 44 and standard deviation 2 and 1.5 respectively. What is the probability that randomly chosen values of X and Y differ by 1.5 or more?

Soln:

Given X is the normal variate with

$\mu_1 = 45$ and $\sigma_1 = 2$

Y is the normal variate with

$\mu_2 = 44$ and $\sigma_2 = 1.5$

By Additive property,

$U = X - Y$ is also random variable with N.D.

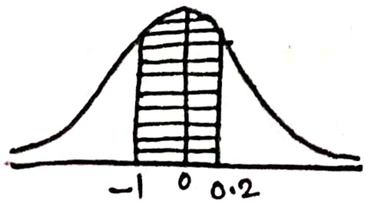
$$E(U) = E(X-Y)$$
$$= E(X) - E(Y)$$
$$= \mu_1 - \mu_2$$
$$= 45 - 44$$
$$\boxed{\mu = 1}$$

$$\text{var}(U) = \text{var}(X-Y)$$
$$= \text{var}(X) + \text{var}(Y)$$
$$= \sigma_1^2 + \sigma_2^2$$
$$= 2^2 + (1.5)^2$$
$$\boxed{\text{Var}(U) = 6.25}$$

$$S.D = \sigma = \sqrt{\text{var}(U)} = \sqrt{6.25} = 2.5$$
$$S.D = \boxed{\sigma = 2.5}$$

To find: $P[|X-Y| \geq 1.5]$

$$P[|X-Y| > 1.5] = 1 - P[|X-Y| \leq 1.5]$$
$$= 1 - P[-1.5 < (X-Y) < 1.5]$$
$$= 1 - P(-1.5 < U < 1.5)$$
$$= 1 - P\left(\frac{-1.5-1}{2.5} < \frac{U-1}{2.5} < \frac{1.5-1}{2.5}\right)$$
$$= 1 - P(-1 < Z < 0.2)$$



$$= 1 - [P(0 < z < 1) + P(0 < z < 0.2)]$$

$$= 1 - [0.3413 + 0.0793]$$

$$= 1 - 0.4206$$

$$= 0.5794$$

7) X is a normal variate with $\mu = 30$ and $\sigma = 5$.

Find (i) $P(26 \leq X \leq 40)$

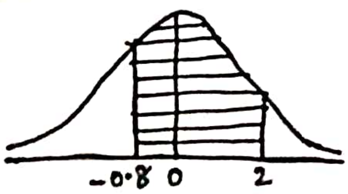
(ii) $P(X \geq 45)$

(iii) $P[|X - 30| \geq 5]$

Soln:

Given $\mu = 30$ and $\sigma = 5$

(i) $P(26 \leq X \leq 40) = P\left[\frac{26-30}{5} \leq \frac{X-30}{5} \leq \frac{40-30}{5}\right]$



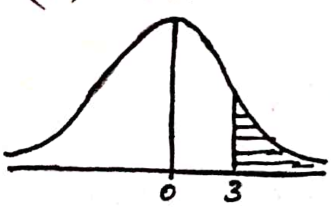
$$= P(-0.8 \leq z \leq 2)$$

$$= P(0 \leq z \leq 0.8) + P(0 \leq z \leq 2)$$

$$= 0.2881 + 0.4772$$

$$= 0.7653$$

(ii) $P(X \geq 45) = P\left(\frac{X-30}{5} \geq \frac{45-30}{5}\right)$



$$= P(z \geq 3)$$

$$= 0.5 - P(0 \leq z \leq 3)$$

$= 0.5 - 0.4987$

$= 0.0013$ //

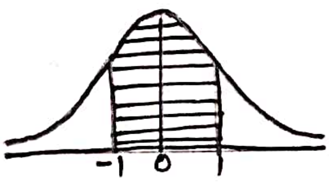
(iii) $P[|X-30|>5] = 1 - P[|X-30| \leq 5]$

$= 1 - P[-5 < (X-30) < 5]$

$= 1 - P[25 < X < 35]$

$= 1 - P\left[\frac{25-30}{5} < \frac{X-30}{5} < \frac{35-30}{5}\right]$

$= 1 - P[-1 < Z < 1]$



$= 1 - 2P(0 < Z < 1)$

$= 1 - 2(0.3413)$

$= 1 - 0.6826$

$= 0.3174$ //

8) Let X and Y be independent normal variates with means 1 and 2 and variances 4 and 3 respectively. Find the distribution of $U = X + 2Y$.

Solution:

Since X and Y are independent normal variates with Mean of $X = \mu_1 = 1$, Var $\sigma_1^2 = 4$

Mean of $Y = \mu_2 = 2$, Var $\sigma_2^2 = 3$

By additive property, the mean of U is

$E(U) = E(X + 2Y)$

$= E(X) + 2E(Y)$

$= 1 + 2(2)$

$\boxed{E(U) = 5}$ //

$Var(U) = Var(X + 2Y)$

$= Var(X) + 4\,Var(Y)$

$= 4 + 4(3)$

$\boxed{Var(U) = 16}$ //

By additive property, U is also a normal variate with mean 5 and variance 16.

9) If X is a normal variate with mean 50 and S.D = 10. Find $P[Y \le 3137]$ where $Y = X^2 + 1$.

Soln:

$P[Y \le 3137] = P[X^2 + 1 \le 3137]$

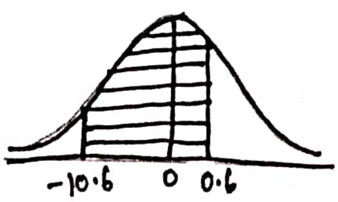
$= P[X^2 \le 3136$

$= P[|X| \le 56]$

$= P[-56 \le X \le 56]$

$= P\left[\frac{-56-50}{10} \le \frac{X-50}{10} \le \frac{56-50}{10}\right]$

$= P[-10.6 \le Z \le 0.6]$



$= P[-10.6 \le Z \le 0] + P[0 \le Z \le 0.6]$

$= P[0 \le Z \le 10.6] + P[0 \le Z \le 0.6]$

$= P(0 \le z \le 3) + P(0 \le z \le 0.6)$

$= 0.5 + 0.2257$

$P(Y \le 3137) = 0.7257$ //

$\left[\because P(0 \le z \le 3) = 0.4987 \simeq 0.5 \quad \therefore P(0 \le z \le 10.6) = 0.5 \right.$

10) The mean yield for one acre plots is 662 kgs with S.D 32. Assuming normal distribution, how many one acre plots in a batch of 1000 plots would you expect to yield.

(i) over 700 kgs

(ii) below 650 kgs

Soln:

X is a random variable denoting the yield (in kgs) for one acre plot.

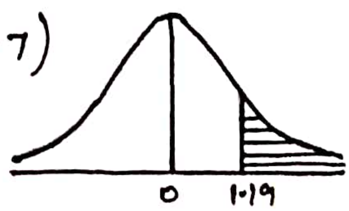
Given $\mu = 662$, $\sigma = 32$.

(i) $P(X > 700) = P\left[\dfrac{X-662}{32} > \dfrac{700-662}{32}\right]$

$= P(z > 1.187)$

$= 0.5 - P(0 < z < 1.187)$

$= 0.5 - P(0 < z < 1.19)$



$= 0.5 - 0.3830$

$= 0.1170$

$\therefore$ The number of plots expected to yield 700 kgs

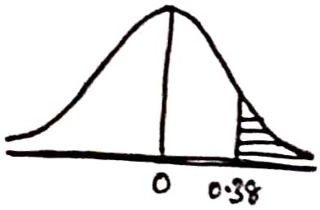
$= 1000 \times 0.1170$

$= 117$ //

(ii) $P(X < 650) = P\left[\frac{X-662}{32} < \frac{650-662}{32}\right]$

$= P[Z < -0.38]$

$= P(Z > 0.38)$



$= 0.5 - P(0 < Z < 0.38)$

$= 0.5 - 0.148$

$= 0.352$ //

The number of plots expected to yield below 650 kgs

$= 1000 \times 0.352$

$= 352$ plots //

11) If M.G.F $= M_X(t) = e^{3t+8t^2}$. find $P(-1 < X < 9)$.

Soln:

For $X \sim N(\mu, \sigma^2)$,

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \rightarrow (1)$$

Given: $M_X(t) = e^{3t+8t^2} \rightarrow (2)$

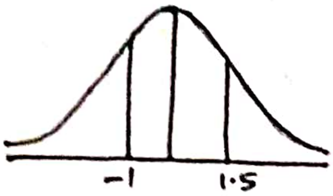
Comparing (1) & (2), we get

$\boxed{\mu = 3}$, $\frac{1}{2}\sigma^2 = 8$

$\Rightarrow \sigma^2 = 8(2) = 16$

$\Rightarrow \boxed{\sigma = 4}$

$P(-1 < X < 9) = P\left(\frac{-1-3}{4} < \frac{X-3}{4} < \frac{9-3}{4}\right)$



$= P\left(-1 < z < \frac{3}{2}\right)$

$= P(0 < z < 1) + P\left(0 < z < \frac{3}{2}\right)$

$= 0.3413 + 0.4332$

$= 0.7745$

12) If x is normally distributed with $\mu = 12$ and $\sigma = 4$. Find the probability of

(i) $x \geq 20$

(ii) $x \leq 20$

(iii) $0 \leq x \leq 12$

(iv) Find a when $P(x > a) = 0.24$

Soln:

Given $\mu = 12$, $\sigma = 4$.

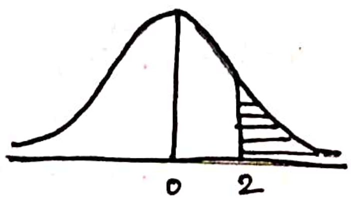
(i) $P(x \geq 20) = P\left[\frac{x-12}{4} \geq \frac{20-12}{4}\right]$

$= P(z \geq 2)$

$= 0.5 - P(0 \leq z \leq 2)$

$= 0.5 - 0.4772$

$= 0.0228$



(ii) $P(x \leq 20) = P\left[\frac{x-12}{4} \leq \frac{20-12}{4}\right]$

$= P[z \leq 2]$

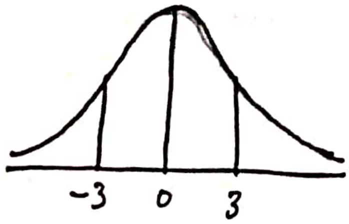
$= 0.5 + P(0 \leq z \leq 2)$

$$= 0.5 + 0.4772$$

$$P(X \le 20) = 0.9772$$

(iii) $$P(0 \le X \le 12) = P\left[\frac{0-12}{4} \le \frac{X-12}{4} \le \frac{12-12}{4}\right]$$

$$= P(-3 \le Z \le 0)$$



$$= P(0 \le Z \le 3)$$

$$= 0.4987$$

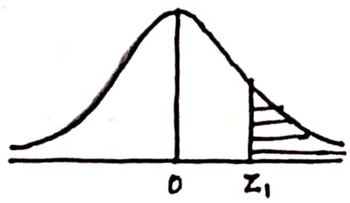
(iv) Given : $P(X > a) = 0.24$

$$P\left[\frac{X-12}{4} > \frac{a-12}{4}\right] = 0.24$$

$$P\left[Z > \frac{a-12}{4}\right] = 0.24$$

$$P(Z > z_1) = 0.24 \qquad \left(\text{Let } \frac{a-12}{4} = z_1\right)$$

$$0.5 - P(0 < Z < z_1) = 0.24$$



$$P(0 < Z < z_1) = 0.5 - 0.24$$

$$P(0 < Z < z_1) = 0.26$$

From the tables, $z_1 = 0.71$

$$\Rightarrow \frac{a-12}{4} = 0.71$$

$$\Rightarrow \boxed{a = 14.84}$$

13) In a Normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and Standard deviation of the distribution. (5.40)

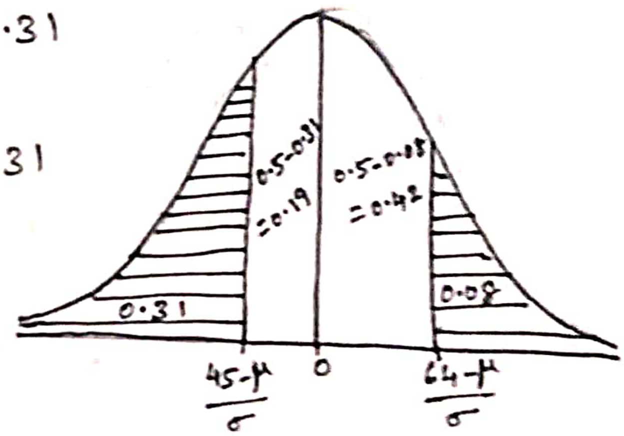
Soln:

Let Mean $= \mu$, S.D $= \sigma$

W.K.T, $Z = \dfrac{X-\mu}{\sigma}$

Here 31% of the items are under 45.

i.e., $P(X < 45) = 0.31$



$\therefore P\left(\dfrac{X-\mu}{\sigma} < \dfrac{45-\mu}{\sigma}\right) = 0.31$

$P\left(Z < \dfrac{45-\mu}{\sigma}\right) = 0.31$

$0.5 - P\left(0 < Z < \dfrac{45-\mu}{\sigma}\right) = 0.31$

$P\left(0 < Z < \dfrac{45-\mu}{\sigma}\right) = 0.5 - 0.31$

$P\left(0 < Z < \dfrac{45-\mu}{\sigma}\right) = 0.19$

Hence $Z = \dfrac{45-\mu}{\sigma} = -0.5 \rightarrow (1)$

(Since, the region is left hand side; So, z must be negative)

Here 8% of the items are over 64.

i.e., $P(X > 64) = 0.08$

$1 - P(X \leq 64) = 0.08$

(5.41)

$$P(X \le 64) = 1 - 0.08 = 0.92$$

$$P\left(\frac{X-\mu}{\sigma} \le \frac{64-\mu}{\sigma}\right) = 0.92$$

$$P\left(Z \le \frac{64-\mu}{\sigma}\right) = 0.92$$

$$0.5 + P\left(0 \le Z \le \frac{64-\mu}{\sigma}\right) = 0.92$$

$$P\left(0 \le Z \le \frac{64-\mu}{\sigma}\right) = 0.92 - 0.5$$

$$P\left(0 \le Z \le \frac{64-\mu}{\sigma}\right) = 0.42$$

Hence, $Z = \dfrac{64-\mu}{\sigma} = 1.41 \rightarrow (2)$

(Since, the region in right hand side; So, Z must be positive)

$$(1) \Rightarrow \frac{45-\mu}{\sigma} = -0.5$$

$$\Rightarrow 45 - \mu = -0.5\sigma$$

$$\boxed{-0.5\sigma + \mu - 45 = 0} \rightarrow (3)$$

$$(2) \Rightarrow \frac{64-\mu}{\sigma} = 1.41$$

$$\Rightarrow 64 - \mu = 1.41\sigma$$

$$\Rightarrow \boxed{1.41\sigma + \mu - 64 = 0} \rightarrow (4)$$

Solving (3) & (4),

$$(3) \Rightarrow -0.5\sigma + \mu - 45 = 0$$

$$(4) \Rightarrow \underset{(-)}{1.41\sigma} \underset{(-)}{+\mu} \underset{(+)}{-64} = 0$$

$$-1.91\sigma \qquad +19 = 0$$

$$-1.91\,\sigma = -19$$

$$\sigma = \frac{19}{1.91} = 9.9476$$

$$\boxed{\sigma \simeq 10}$$

Substitute $\sigma = 10$ in (3),

$$-0.5(10) + \mu - 45 = 0$$

$$-5 + \mu - 45 = 0$$

$$\mu - 50 = 0$$

$$\boxed{\mu = 50}$$

H.W.

14) In a distribution exactly normal, 7 % of items are under 35 and 89 %. are under 63. What are the mean and S.D of the distribution.

[Ans: Mean = 50.29 , S.D = 10.33]